MATH 11 FALL 2008: LECTURE 8

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1. Implicit differentiation continued

Example 1 (Power rule revisited)**.** *Let $y = x^{m/n}$. Then $y^n = x^m$. Thus,*

$$ny^{n-1}\frac{dy}{dx} = mx^{m-1}.$$

So we compute,

$$\begin{aligned}\frac{dy}{dx} &= \frac{mx^{m-1}}{ny^{n-1}} \\ &= \frac{m}{n}\cdot\frac{x^m}{y^n}\cdot\frac{x^{-1}}{y^{-1}} \\ &= \frac{m}{n}\cdot\frac{y}{x} \\ &= \frac{m}{n}\cdot\frac{x^{m/n}}{x} \\ &= \frac{m}{n}\cdot x^{m/n-1}.\end{aligned}$$

Integration

Our motivation for studying integration is geometric: what is the area under the graph of a curve $y = f(x)$? If f is constant this is easy to study. Similarly, if f is linear, the area is easily computed (it's a trapezoid).

So, the state of our knowledge seems to be: if a function $f(x)$ is very nice, like a line, then we can compute the area A under f between two points without difficulty. And we're stuck otherwise. So, the idea is to build off of what we know. Let's first *approximate* the area under the graph by comparing it to the area of a bunch of rectangles (because we know how to compute the area of rectangles). The bases of our rectangles will be non-overlapping portions of the interval from a to b, which all together cover all of $[a, b]$. Let's make this precise.

Definition 2. *A* partition *P of the interval $[a, b]$ is a collection of subintervals*

$$\begin{aligned}I_1 &= [x_0, x_1] = [a, x_1] \\ I_2 &= [x_1, x_2] \\ &\vdots \\ I_n &= [x_{n-1}, x_n] = [x_{n-1}, b].\end{aligned}$$

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Note that $a = x_0$ and $b = x_n$ in the above. The length of the kth subinterval is denoted $\triangle x_k$ and is given by

$$|I_k| = \triangle x_k = x_k - x_{k-1}.$$

A partition of $[a, b]$ provides bases for our rectangles. In order approximate the area under the graph, let's find a collection of rectangles whose total area is *less* than the area A, and another collection whose total area is more than A. This can be directly accomplished by considering rectangles whose heights are either the minimum or maximum of the value of f on a given subinterval.

Definition 3.

$$M_k = \mathit{Min}\{f(x) : x \in I_k\}$$
$$m_k = \mathit{Max}\{f(x) : x \in I_k\}$$

Therefore, the area under the graph of $y = f(x)$ over the interval $[x_{k-1}, x_k]$ is at least $m_k \cdot \triangle x_k$ and at most $M_k \cdot \triangle x_k$. To compare to A, we should sum over all subintervals.

Definition 4. *The* lower Riemann sum of f with respect to the partition P *is*

$$L(f, P) = \sum_{k=1}^{n} m_k \cdot \triangle x_k,$$

and the upper Riemann sum of f with respect to the partition P *is given by*

$$U(f, P) = \sum_{k=1}^{n} M_k \cdot \triangle x_k.$$

Now we have an approximation of the area A. By construction, we have

$$L(f, P) \leq A \leq U(f, P)$$

for any partition P of the interval $[a, b]$. We are now ready to define integration.

Definition 5. *If there is only one number A such that*

$$L(f, P) \leq A \leq U(f, P)$$

for <u>every</u> partition P of $[a, b]$, then we say that f is integrable *on $[a, b]$. The definite* integral *of f from a to b is this number A and is denoted by*

Proposition 6. *Let P_n denote the* regular *partition of $[a, b]$, whose n subintervals all have equal length. Then A is integrable if and only if*

$$\lim_{n\to\infty} L(f, P_n) = A = \lim_{n\to\infty} U(f, P_n).$$

Proof. We leave the proof as a thought experiment for the curious reader.

Example 7. *Find*

$$\int_0^b x^2 dx$$

using the definition of the derivative (or it's equivalent reformulation in Proposition 6).

Solution. If we choose n equally spaced subintervals covering $[a, b]$, each one must have length

$$\triangle x_k = \frac{b-a}{n} = \frac{b}{n}.$$

Thus the endpoints of our subintervals are

$$\begin{aligned}
x_0 &= a = 0 \\
x_1 &= x_0 + b/n = 0 + b/n = b/n \\
x_2 &= x_1 + b/n = 2b/n \\
&\vdots \\
x_k &= kb/n \\
&\vdots \\
x_n &= nb/n = b.
\end{aligned}$$

So we compute

$$\begin{aligned}
U(f, P_n) &= \sum_{k=1}^{n} m_k \cdot \triangle x_k \\
&= \sum_{k=1}^{n} f(x_k) \triangle x_k \\
&= \sum_{k=1}^{n} (x_k)^2 \frac{b}{n} \\
&= \sum_{k=1}^{n} \left(\frac{kb}{n}\right)^2 \cdot \frac{b}{n} \\
&= \frac{b^3}{n^3} \sum_{k=1}^{n} k^2 \\
&= \frac{b^3}{n^3} \cdot \frac{n(n+1)(2n+1)}{6} \\
&= \frac{b^3}{6} \left(1 + \frac{1}{n}\right) \left(2 + \frac{1}{n}\right).
\end{aligned}$$

Therefore

$$\lim_{n \to \infty} U(f, P_n) = \frac{b^3}{3}.$$

A similar calculation shows

$$\lim_{n \to \infty} L(f, P_n) = \frac{b^3}{3}.$$

Therefore, x^2 is integrable over $[0, b]$ and

$$\int_0^b x^2 dx = \frac{b^3}{3}.$$