Math 171 - Abstract Algebra I Section _
HW #11
10/16/08

Dummit & Foote (7.3) 10, 13, 26, 29

7.3.10 Decide which of the following are ideals of the ring $\mathbb{Z}[x]$:

(a) the set of all polynomials whose constant term is a multiple of 3

(b) the set of all polynomials whose coefficient of x^2 is a multiple of 3

(c) the set of all polynomials whose constant term, coefficient of x and coefficient of x^2 are zero

(d) $\mathbb{Z}[x^2]$ (i.e., the polynomials in which only even powers of x appear)

(e) the set of polynomials whose coefficients sum to zero

(f) the set of polynomials $p(x)$ such that $p'(0) = 0$, where $p'(x)$ is the usual first derivative of $p(x)$ with respect to x.

7.3.13 Prove that the ring $M_2(\mathbb{R})$ contains a subring that is isomorphic to $\mathbb{C}$.

7.3.26 The *characteristic* of a ring R is the smallest positive integer n such that $1+1+\cdots+1=0$ (n times) in R; if no such integer exists the characteristic of R is said to be 0. For example, $\mathbb{Z}/n\mathbb{Z}$ is a ring of characteristic n for each positive integer n and $\mathbb{Z}$ is a ring of characteristic 0.

(a) Prove that the map $\mathbb{Z} \to R$ defined by

$$k \mapsto \begin{cases} 1+1+\cdots+1 \ (k \text{ times}) & \text{if } k>0 \\ 0 & \text{if } k=0 \\ -1-1-\cdots-1 \ (k \text{ times}) & \text{if } k<0 \end{cases}$$

is a ring homomorphism whose kernel is $n\mathbb{Z}$, where n is the characteristic of R (this explains the use of the terminology "characteristic 0" instead of the archaic phrase "characteristic ∞" for rings in which no sum of 1's is zero).

(b) Determine the characteristics of the rings $\mathbb{Q}$, $\mathbb{Z}[x]$, $\mathbb{Z}/n\mathbb{Z}[x]$.

(c) Prove that if p is a prime and if R is a commutative ring of characteristic p, then $(a+b)^p = a^p + b^p$ for all $a, b \in R$.

7.3.29 Let R be a commutative ring. Recall (cf. Exercise 13, Section 1) that an element $x \in R$ is nilpotent if $x^n = 0$ for some $n \in \mathbb{Z}^+$. Prove that the set of nilpotent elements form an ideal — called the *nilradical* of R and denoted by $\mathfrak{N}(R)$. [Use the Binomial Theorem to show $\mathfrak{N}(R)$ is closed under addition.]