1 Isomorphism Theorems, Continued

1.1 The First Isomorphism Theorem

Recall: The First Isomorphism Theorem or The Homomorphism Theorem:
Let \( \varphi : G \to G' \) be a homomorphism of groups. Then
\[
G / \ker(\varphi) \cong \Im(\varphi).
\]
Note that this implies a) \( \ker(\varphi) \) is a normal subgroup of \( G \), and b) \( \Im(\varphi) \) is a group.

Defn: In fact, there exists an isomorphism \( \theta : G / \ker(\varphi) \to \Im(\varphi) \) such that this diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\varphi} & G' \\
\downarrow{\pi} & \circ & \downarrow{\iota} \\
G / \ker(\varphi) & \xrightarrow{\theta} & \Im(\varphi)
\end{array}
\]

\( \varphi = \iota \circ \theta \circ \pi \)

In other words, \( \forall x \in G, \varphi(x) = \iota(\theta(\pi(x))) \).

Ex: Let \( G \) be a group. Let \( a \in G \) be any element. If \( a \) has infinite order, then \( \langle a \rangle \cong \mathbb{Z} \) (under addition, of course). If \( a \) has finite order \( n \), then \( \langle a \rangle \cong \mathbb{Z}/n\mathbb{Z} \).

Proof: Consider the map \( \pi : \mathbb{Z} \to G \) defined by \( \pi(k) = a^k \). If \( a \) has infinite order, i.e. there does not exist \( k \neq 0 \) such that \( a^k = 1 \), then \( \ker(\pi) = \{0\} \). Note that \( \Im(\pi) = \{a^k \mid k \in \mathbb{Z}\} = \langle a \rangle \). But by the First Isomorphism Theorem, we know that \( \mathbb{Z} / \ker(\pi) \cong \Im(\pi) \), so \( \mathbb{Z} \cong \langle a \rangle \) (because \( \mathbb{Z}/\{0\} \cong \mathbb{Z} \), and \( \Im(\pi) = \langle a \rangle \)). If \( a \) has finite order, i.e. \( |a| = n \). Inspect \( \ker(\pi) \). We have \( \ker(\pi) = \{k \in \mathbb{Z} \mid a^k = 1 \in G\} = n\mathbb{Z} \). So, by the First Isomorphism Theorem, \( \mathbb{Z} / \ker(\pi) = \mathbb{Z}/n\mathbb{Z} = \mathbb{Z}_n \cong \langle a \rangle \).  

\[\blacksquare\]
Ex: Consider the unit circle in the complex plane \((S^1)\). Recall that we can “coil” the real line along this unit circle using the sine and cosine functions. Let us make this relationship more concrete. Consider the map \(\varphi : \mathbb{R} \to S^1\) given by \(t \mapsto e^{it}\). Then \(\text{Im}(\varphi) = S^1\), and \(\ker(\varphi) = \{t \in \mathbb{R} \mid \varphi(t) = 1\} = \langle 2\pi \rangle = 2\pi\mathbb{Z}\). By the First Isomorphism Theorem, \(\mathbb{R}/\langle 2\pi \rangle \cong S^1\).

1.2 The Third Isomorphism Theorem

Motivation: **Theorem:** Let \(N \trianglelefteq G\). Every subgroup of \(G/N\) is of the form \(H/N\), for some unique subgroup \(H \leq G\) containing \(N\). Furthermore, \(H/N \trianglelefteq G/N\) if and only if \(H \leq G\).

**Remark:** Let \(\varphi : G \to G'\) be a group homomorphism. Let \(H' \leq G'\). Then \(\varphi^{-1}(H') \leq G\). Also, \(H \leq G\), then \(\varphi(H) \subseteq G'\). “This sets up a bijection between subgroups on the one hand side, and subgroups on the other hand side.” Consider \(\pi : G \to G/N\). Let \(A \leq G/N\). Study \(\pi^{-1}(A) \leq G\). \(N = 1 \in G\). \(\pi^{-1}(1) \subseteq \pi^{-1}(A)\) because \(A\) is a group. So \(\pi^{-1}(N) \subseteq \pi^{-1}(A)\). \(\ker(\pi) \subseteq \pi^{-1}(A)\). Thus \(N \subseteq \pi^{-1}(A)\).

**Theorem:** Let \(G\) be a group, \(A \trianglelefteq G, B \trianglelefteq G\). If \(A \subseteq B\), then \(A \trianglelefteq B\), and \(B/A \trianglelefteq G/A\), and

\[
(G/A)/(B/A) \cong G/B.
\]

In fact, there exists an isomorphism \(\theta : (G/A)/(B/A) \to G/B\) such that this diagram commutes:

\[
\begin{array}{ccc}
G & \xrightarrow{\pi} & G/A & \xrightarrow{\sigma} & (G/A)/(B/A) \\
& \searrow \circ \swarrow \theta & \downarrow \circ \swarrow \\
& \quad G/B & \quad \quad \quad \quad \quad \quad \quad \\
\end{array}
\]

where \(\pi, \rho, \sigma\) are projections. In other words, \(\rho = \theta \circ \sigma \circ \pi\).

**Proof:** Define \(\omega : G/A \to G/B\) by \(Ax \mapsto Bx\). By the First Isomorphism Theorem, \(B/A \trianglelefteq G/A\) by inspection of the kernel. \(\ker(\omega) = \{Ax \in G/A \mid Bx = B\} = \{Ax \in G/A \mid x \in B\} = B/A\). \((G/A)/\ker(\omega) \cong \text{Im}(\omega)\), so \((G/A)/(B/A) \cong G/B\). \(\blacksquare\)
1.3 The Second Isomorphism Theorem

Theorem: Let $G$ be a group, $A \leq G$, $N \trianglelefteq G$. Then $AN \leq G$, and $N \trianglelefteq AN$, and $A \cap N \trianglelefteq A$, and $AN/N \cong A/(A \cap N)$. In fact, there exists an isomorphism $\theta : AN/N \to A/(A \cap N)$ such that the following diagram commutes:

$$
\begin{array}{ccc}
A & \xrightarrow{l} & AN \\
\downarrow{\pi} & \circ & \downarrow{\rho} \\
A/(A \cap N) & \xleftarrow{\theta} & AN/N
\end{array}
$$

Ex: Consider $GL_n(\mathbb{C})$, the group of invertible linear transformations from $\mathbb{C}^n \to \mathbb{C}^n$. Consider $SL_n(\mathbb{C})$, maps with determinant 1. Then $GL_n(\mathbb{C})/SL_n(\mathbb{C}) \cong \mathbb{C}^\times$. 
