0.1. **Exercise.** Let R be any ring and $n \geqslant 2$. Show that $M_n(R)$ is not commutative by considering the following example. Let $a, b \in R$ be such that $ab \neq 0$ and let

$$A = \begin{pmatrix} a & 0 & \\ 0 & 0 & \\ & & \ddots \end{pmatrix} \qquad B = \begin{pmatrix} 0 & b & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \ddots \end{pmatrix}$$

Compare AB and BA.

Definition 1. *The units of* $M_n(R)$ *are called* the general linear group, *which is denoted*

$$GL_n(R) = \{A \in M_n(R) : A \textit{ is invertible}\}$$

Definition 2. *Let* R *be a commutative ring with identity.*

(1) *The* special linear group, $SL_n(R)$, *is the subgroup of* $GL_n(R)$ *consisting of matrices with determinant* 1.
(2) *The* orthogonal group, $O_n(R)$, *is the subgroup of* $GL_n(R)$ *consisting of orthogonal matrices, i.e. those matrices* A *such that* $AA^t = I$, *where* A^t *is the transpose of* A, *and* I *is the identity matrix.*
(3) *The* unitary group, U_n, *is the subgroup of* $GL_n(\mathbb{C})$ *of matrices such that* $AA^* = I$, *where* A^* *is the transpose conjugate of* A.
(4) *The* symplectic group Sp_n *is the subgroup of* $GL_n(\mathbb{H})$ *such that* $AA^* = I$, *where* A^* *is the quaternionic complex conjugate of* A.

1. Group Rings

Definition 3. *Let* R *be a commutative ring with identity and let* G *be a finite group*

$$G = \{g_1, \dots g_n\}.$$

The group ring RG *of* G *with coefficients in* R *is the set of all formal sums*

$$RG = \{a_1 g_1 + \cdots a_n g_n : a_i \in R,\ g_i \in G\}.$$

If e is the identity in G and $a \in R$, we define

$$a \cdot e = a \in RG.$$

Similarly, for the identity $1 \in R$ and $g \in G$ we define

$$1 \cdot g = g \in RG.$$

Addition is defined componentwise

$$(a_1 g_1 + \cdots + a_n g_n) + (b_1 g_1 + \cdots + b_n g_n) = ((a_1 + b_1) g_1 + \cdots + (a_n + b_n) g_n).$$

Multiplication is defined by

$$(a g_i) \cdot (b g_j) = (ab)(g_i g_j),$$

with the additional requirement that the distributive law holds.

1.1. **Exercise.** Show that RG has zero divisors for any non-trivial group G by considering the product

$$(1 - g)(1 + g + g^2 + \cdots + g^{m-1}),$$

where $g \in G$ is an element of order m.

1.2. **Exercise.** Show that $\mathbb{R}Q$ is not the same ring as $\mathbb{H}$, where Q is the quaternion group and $\mathbb{H}$ are is the quaternion ring.