MATH 171 FALL 2008: LECTURE 18

DAGAN KARP

1. GROUP ACTIONS REVISITED

Recall a *group action* of the group G on the set S is a map of sets

$$G \times S \to S$$

such that for all $x, y \in G$ and $s \in S$

$$x \cdot (y \cdot s) = (xy) \cdot s$$

and

$$1 \cdot s = s.$$

This gives rise to the notion of permutation representation.

Definition 1. *Let G be a group and A a set. A homomorphism from G to the group of permutations of A*

$$\sigma : G \to S_A$$

is called a permutation representation *of G. We then call A a G-set.*

Remark 2. Note that a group action gives rise to (induces a) permutation representation and conversely. In other words, there is a one to one correspondence between group actions and permutation representations. Indeed, given a group action of G on A, we define a homomorphism

$$\sigma : G \to S_A$$

by

$$\sigma(x) \mapsto \sigma_x,$$

where $\sigma_x \in S_A$ is defined by

$$\sigma_x(a) = x \cdot a$$

for any $a \in A$. That σ is a homomorphism follows directly from the properties of group actions:

$$\sigma(xy)(a) = \sigma_{xy}(a) = (xy) \cdot a = x \cdot (y \cdot a) = \sigma_x(y \cdot a) = \sigma_x(\sigma_y(a)) = \sigma(x)(\sigma(y)(a)).$$

Conversely, given a permutation representation of G with set A,

$$\sigma : G \to S_A,$$

Date: November 10, 2008.

we can construct a map

$$G \times A \to A$$

given by

$$(x, a) \mapsto \sigma(x)(a) = x \cdot a.$$

That this is a group action follows directly from the fact that σ is a homomorphism:

$$(xy) \cdot a = \sigma(xy)(a) = \sigma(x) \circ \sigma(y)(a) = x \cdot (y \cdot a),$$

and

$$1 \cdot a = \sigma(1)(a) = I(a) = a,$$

where $I \in S_A$ is the identity permutation, and the second to last equality holds because identity maps to identity under any homomorphism.

Example 3.

(1) *The group* S_n *of permutations of* n *elements acts on the set* $\{1, \dots, n\}$*.*
(2) *Any group acts on any set trivially, setting* $g \cdot s = s$ *for all* $(g, s) \in G \times S$*.*
(3) *The dihedral group* D_4 *acts on the set of vertices of the square.*
(4) *The general linear group* $GL_n(\mathbb{R})$ *acts on* $\mathbb{R}^n$ *by matrix multiplication*

$$A \cdot (x_1, \dots, x_n) = \begin{pmatrix} a_{1,1} & \cdots & a_{1,n} \\ \vdots & & \vdots \\ a_{n,1} & \cdots & a_{n,n} \end{pmatrix} \times \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}$$

1.1. **Conjugation.** The group G may act on itself, that is act on the set of elements of G. For $x \in G$, let

$$c_x : G \to G$$

be defined by

$$c_x(y) \mapsto xyx^{-1}.$$

This gives rise to a permutation representation. In fact, we have a homomorphism

$$G \to \mathrm{Aut}(G).$$

Indeed

$$c_x(yz) = xyzx^{-1} = (xyx^{-1})(xzx^{-1}) = c_x(y)c_x(z).$$

Further, the map $c_{x^{-1}} : G \to G$ is immediately seen to be an inverse of c_x.

Definition 4. *The above group action* $G \times G \to G$ *given by* $(x, g) \mapsto c_x g$ *is called* conjugation*. By abuse of terminology, the associated permutation representation is also called conjugation.*

Definition 5. *For each* $x \in G$*, the automorphism* c_x *is called an* inner *automorphism of* G*.*

Definition 6. *Let* $G \to \mathrm{Aut}(G)$ *be be given by conjugation. The kernel of this homomorphism is called the* center *of* G*, denoted*

$$Z(G) = \{x \in G : c_x = I\},$$

where $I \in \mathrm{Aut}(G)$ *is the identity automorphism.*

Remark 7. We have

$$\begin{aligned}Z(G) &= \{x \in G : c_x = I\}\\ &= \{x \in G : xyx^{-1} = y \text{ for all } y \in G\}\\ &= \{x \in G : xy = yx \text{ for all } y \in G\}.\end{aligned}$$

So the center of G is the set of all elements which commute with all elements of G.

Definition 8. *Let* A *be a* G*-set, and let* $a \in A$*. The* stabilizer *of the element* a *is the set of elements of* G *which fix* a*, denoted* G_a*,*

$$G_a = \{x \in G : x \cdot a = a\}.$$

Definition 9. *The* centralizer *of* $x \in G$ *is the stabilizer of the element* x *under the action of conjugation*

$$C(x) = C_G(x) = \{y \in G : c_y(x) = x\}.$$

Remark 10. Note that

$$C(x) = \{y \in G : c_y(x) = x\} \tag{1}$$

$$= \{y \in G : yxy^{-1} = x\} \tag{2}$$

$$= \{y \in G : yx = xy\}. \tag{3}$$

Hence, the centralizer of x is the set of all elements of G commuting with x.

Proposition 11. *When the group* G *acts on the set* A*, the binary relation*

$$a \sim b \iff a = x \cdot b \text{ for some } x \in G$$

is an equivalence relation on A*.*

Proof. $1 \cdot x = x$, so the relation is reflexive. It is symmetric, as $a = x \cdot b$ implies $b = x^{-1} \cdot a$. It is transitive, as $a = x \cdot b$ and $b = y \cdot c$ imply

$$a = x \cdot (y \cdot c) = (xy) \cdot c.$$

□

Definition 12. *Let* G *act on* A *and* $a \in A$*. The* orbit *of* a *under the action of* G *is the equivalence class of* a *under the above equiv lance relation.*

Definition 13. *The action of the group* G *on the set* A *is* transitive *if there is only one orbit, i.e, given* $a, b \in A$ *there exists* $x \in G$ *such that* $a = x \cdot b$*.*

Example 14. *The group* S_n *acts transitively on* $\{1, 2, \ldots, n\}$*.*

Proposition 15. *The stabilizer* G_a *of* a *is a subgroup of* G*. There is a one to one correspondence between the left cosets of* G_a *and the elements of the orbit of* a*; hence the order of the orbit of* a *equals the index of* G_a*.*

Proof. If $x, y \in G_a$, then $a = y \cdot a$, hence $y^{-1} \cdot a = a$ and

$$xy^{-1} \cdot a = x \cdot (y^{-1} \cdot a) = x \cdot a = a.$$

Thus $xy^{-1} \in G_a$ and therefore $G_a \leqslant G$.

Now, let b be an element of the orbit of a, i.e. $b = x \cdot a$ for some $x \in G$. Define

$$C(b) = \{y \in G : y \cdot a = b\}.$$

Then $y \in C(b)$ if and only if $y \cdot a = x \cdot a$, and so

$$a = x^{-1} \cdot (y \cdot a) = (x^{-1}y) \cdot a.$$

Therefore $x^{-1}y \in G_a$ and $y \in xG_a$. Therefore $C(b) = xG_a$ is a left coset of G_a.

Conversely, let xG_a be a left coset of G_a. Then $y \in xG_a$ if and only if $x^{-1}y \in G_a$, i.e. $x^{-1}y \cdot a = a$, i.e., $y \cdot a = x \cdot a$. Hence $xG_a \cdot a$ is a single element of A; call it $\theta(xG_a)$.

The maps θ and C are mutually inverse bijections.

Corollary 16. *For each $x \in G$, the centralizer $C_C(x)$ is a subgroup of G, and the number of conjugates of x equals the index of the centralizer*

$$|C_x| = [G : C_G(x)].$$