MATH 171 FALL 2008: LECTURE 23

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In this lecture, we'll begin our study of modules. This lecture is largely based on Lang and Dummit&Foote.

1. Introduction to Modules

Definition 1. *Let R be a ring, and M an Abelian group. We say that M is a* (left) R-module *(or a* module over R*) if there is a multiplicative (left) action of R on M such that, for any $a, b \in R$ and $x, y \in M$, we have*

$$a \cdot (x + y) = a \cdot x + a \cdot y \qquad (a + b) \cdot x = a \cdot x + b \cdot x.$$

Remark 2. By definition of action, we have

$$(ab) \cdot x = a \cdot (b \cdot x) \qquad 1 \cdot x = x,$$

where the latter holds for rings with unity. Note also that $a(-x) = -ax$ and that $0x = 0$.

Definition 3. *Let M be an R-module. An additive subgroup $N \leqslant M$ is a* submodule *of M if $R \cdot N \subset N$, i.e., for all $a \in R$, $n \in N$,*

$$a \cdot n \in N.$$

Remark 4. A submodule N of the R-module M is again an R-module, with action induced by that of R on M.

Example 5. *Let R be any ring.*

1. *Any ring R is itself an R-module.*
2. *The zero group $\{0\}$ is an R-module for any ring R.*
3. *Any Abelian group is a $\mathbb{Z}$-module.*

Remark 6. Note that every Abelian group is a $\mathbb{Z}$-module and conversely. Similarly for subgroups.

Definition 7. *A* vector space *is a module over a field.*

Example 8. *Let F be a field, and let $F^n = \bigoplus_{i=1}^n F$. Then F^n is a vector space with elements*

$$(a_1, \ldots, a_n) \in F^n$$

where $a_i \in F$ for all i. The addition and scalar multiplication are defined componentwise.

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Remark 9. Let V be a vector space over the field F. Recall that a linear map, or a linear transformation, is a map of sets $L : V \to V$ such that, for all $a, b \in F$ and $u, v \in V$,

$$L(au + bv) = aL(u) + bL(v).$$

Example 10. *Let V be a vector space over the field F, and let R be the set of all linear maps from V to itself. Then V is an R-module.*

Example 11. *Let S be a non-empty set and M an R-module. Then the set of maps Map(S, M) is an R-module. We've already seen that this is an Abelian group. To see the module structure, for $a \in R$ and $f : S \to M$, define $a \cdot f$ to be the map such that*

$$(a \cdot f)(s) = a \cdot (f(s)).$$

Definition 12. *Let R be a commutative ring with unity, and let M be an R-module. The* torsion submodule M_{tor} *is given by*

$$M_{tor} = \{x \in M : a \cdot x = 0 \textit{ for some } 0 \neq a \in R\}.$$

Definition 13. *Let N be a submodule of M over R. The* annihilator *of N is the set*

$$\{a \in R : a \cdot x = 0 \textit{ for all } x \in M\}.$$

Remark 14. The annihilator of a submodule is an ideal of R, and M_{tor} is a submodule of M.

2. Basic properties of modules

Let M be an R module, and I, J ideals of R. Define IM by

$$IM = \{a_1 x_1 + \cdots + a_n x_n : a_i \in R,\ x_i \in M,\ n \in \mathbb{N}\}.$$

Then IM is a submodule of M. Note that we have associativity

$$(IJ)M = I(JM)$$

and also distributivity

$$(I + J)M = IM + JM.$$

Further, if N, N' are submodules of M, then

$$I(N + N') = IN + IN'.$$

Definition 15. *Let M be an R-module and N a submodule. Define the* quotient module (or factor module) *of M by N to be the set of cosets M/N with R action given by, for any $a \in R$ and $x + N \in M/N$,*

$$a \cdot (x + N) = a \cdot x + N.$$

Definition 16. *Let $M.M'$ be R-modules. A* module homomorphism $f : M \to M'$ *is an additive group homomorphism such that*

$$f(a \cdot x) = a \cdot f(x)$$

for all $a \in R$ and $x \in M$.

Remark 17. R-module homomorphisms are also called R-homomorphisms or R-linear maps.

Definition 18. *An invertible module homomorphism is called a* module isomorphism*.*

Example 19. *Let* M *and* M' *be modules.*

(1) *The zero map* $\zeta : M \to M'$ *is a module hom.*
(2) *The identity map is a module hom.*
(3) *For any submodule* $N \leqslant M$*, the projection map* $M \to M/N$ *is a module hom.*

Theorem 20 (Module Isomorphism Theorems)**.** *Let* M, M' *be* R*-modules, and let* A, B *be submodules of* M*.*

(1) *Let* $f\phi : M \to M'$ *be an* R*-hom. Then* $\operatorname{Ker}\phi$ *is a submodule of* M *and*

$$M/\operatorname{Ker}\phi \cong \phi(M)/$$

(2)

$$(A + B)/B \cong A/(A \cap B)$$

(3) *If* $A \subseteq B$*, then*

$$(M/A)/(B/A) \cong M/B.$$

(4) *There exists a bijection between submodules of* M/A *and submodules* N *of* M *containing* A*. The correspondence is given by*

$$N \Longleftrightarrow N/A$$

for all $A \subset N$*.*