MATH 171 FALL 2008: LECTURE 24

DAGAN KARP

1. Modules continued

Let V be a vector space over the field K, and $T : V \to V$ a linear transformation. Then V is endowed with the structure of a $K[x]$ module as follows. To define the action of $K[x]$ on V, let $f(x) = a_0 + a_1 x + \cdots + a_n x^n \in K[x]$ and $v \in V$. Then define

$$f(x) \cdot v = \sum_{i=0}^{n} a_i T^i(v),$$

where T^i is T composed with itself i-times, and $T^0 = I$, the identity map on V.

To see that this is an action, note that

$$1 \cdot v = 1T^0(v) = I(v) = v,$$

and, for f as above and $g(x) = \sum_{j=1}^{m} b_j x^j \in K[x]$ and $v \in V$ we have

$$\begin{aligned} f(x) \cdot (g(x) \cdot v) &= f(x) \cdot \left(\sum_{j=0}^{m} b_j T^j(v) \right) \\ &= \sum_{i=0}^{n} a_i T^i \left(\sum_{j=0}^{m} b_j T^j(v) \right) \\ &= \sum_{i+j=0}^{n+m} a_i b_j T^{i+j}(v) \\ &= (f(x)g(x)) \cdot v. \end{aligned}$$

WLOG assume $n \leqslant m$ and let $a_i = 0$ for $i > n$. Then we compute

$$\begin{aligned} (f(x) + g(x))(v) &= \left(\sum_{k=0}^{m} (a_k + b_k) x^k \right) \cdot v \\ &= \sum_{k=0}^{m} (a_k + b_k) T^k(v) \\ &= \sum_{k=0}^{m} a_k T^k(v) + \sum_{k=0}^{m} b_k T^k(v) \\ &= f(x) \cdot v + g(x) \cdot v. \end{aligned}$$

Date: December 3, 2008.

Finally, we have, for $u, v \in V$ and $f \in K[x]$ as above,

$$\begin{aligned} f(x)\cdot(u+v) &= \sum_{i=0}^{n} a_i T^i(u+v) \\ &= \sum_{i=0}^{n} a_i (T^i(u) + T^i(v)) \\ &= \sum_{i=0}^{n} a_i T^i(u) + \sum_{i=0}^{n} a_i T^i(v) \\ &= f(x)\cdot(u) + f(x)\cdot(v). \end{aligned}$$

Thus, together a vector space V over K and a linear transformation T provide the data for a $K[x]$-module. But the converse is true as well. Given any $K[x]$-module W, it is immediate that W is also a K-module, as $K \subset K[x]$. In other words W is a vector space over K. Moreover, we define a linear transformation $T : W \to W$ via the action of $x \in K[x]$. Define $T : W \to W$ by

$$T(w) = x \cdot w$$

for any $w \in W$. Then for $a, b \in K$ and $w, w' \in W$,

$$\begin{aligned} T(aw + bw') &= x \cdot (aw + bw') \\ &= x \cdot (aw) + x \cdot (bw') \\ &= (xa)\cdot(w) + (xb)\cdot(w') \\ &= ax \cdot w + bx \cdot w' \\ &= aT(w) + bT(w'). \end{aligned}$$

Thus T is indeed a linear transformation. Since the action of K and x on V uniquely determine the action of any element of $K[x]$ on V, we have a bijection

$$\{\text{ K[x]-modules }\} \longleftrightarrow \{\text{ vector spaces over K together with a linear transformation }\}$$

2. Classification of all modules: the fundamental theorem

Any finitely generated module over a PID is isomorphic to a direct sum of finitely many cyclic modules.

Theorem 1 (Fundamental Theorem of Finitely Generated Modules)**.** *Let M be a finitely generated module over the PID R. Then*

$$M \cong R^r \oplus R/(p_i^{\alpha_1}) \oplus R/(p_2^{\alpha_2}) \oplus \cdots \oplus R/(p_l^{\alpha_l}),$$

where $r, l \geqslant 0$ are a non-negative integers and $p_i^{\alpha_i}$ are positive powers of not necessarily distinct primes in R.

Corollary 2 (Fundamental Theorem of Finitely Generated Abelian Groups)**.** *Let* G *be a finitely generated Abelian group. Then*

$$G \cong \mathbb{Z}^r \oplus \mathbb{Z}_{p_1^{n_1}} \oplus \mathbb{Z}_{p_2^{n_2}} \oplus \cdots \oplus \mathbb{Z}_{p_l^{n_l}},$$

for some non negative integers r, l *and* n_i *and prime numbers* p_i*.*

Corollary 3. *Let* V *be a finite dimensional vector space over* K*, and let* T *be a linear transformation. If* K *contains all of the eigenvalues of* T*, then there exists a basis of* V *with respect to which the matrix corresponding to* T *is in Jordan canonical form.*