MATH 171 FALL 2008: LECTURE 8

DAGAN KARP

ABSTRACT. In today's lecture we'll be introduced to and discuss the isomorphism theorems. Along the way we'll formally meet several new concepts, including direct products of groups, inner automorphisms, conjugacy classes and the centralizer of a group.

1. THE ISOMORPHISM THEOREMS

Given a homomorphism $\phi : G \to G'$, how far is it from being an isomorphism? The difference between homomorphism and isomorphism is bijectivity. So, well, ϕ may not be surjective. But, any map is onto its *image* by definition, so we may restrict our attention there. Also, our given homomorphism may not be injective. As we have seen, this is equivalent to the statement that $\mathrm{Ker}(\phi)$ may be non-trivial. Well, $\mathrm{Ker}(\phi)$ is a normal subgroup of G (always), and the resulting quotient group $G/\mathrm{Ker}(\phi)$ has the effect of trivializing the kernel!

Theorem 1 (The First Isomorphism Theorem, AKA the Homomorphsim Theorem)**.** *Let $\phi : G \to G'$ be a homomorphism of groups. Then $\mathrm{Ker}\,\phi \trianglelefteq G$, $\mathrm{Im}(\phi) \leqslant G'$ and*

$$G/\mathrm{Ker}\,\phi \cong \mathrm{Im}(\phi).$$

Before we prove this result, let's take a look at an example.

Example 2. *Consider the map*

$$\phi : \mathbb{Z} \to \mathbb{Z}_2,$$

given by

$$n \mapsto n \mod 2.$$

Then

$$\begin{aligned}\mathrm{Ker}(\phi) &= \{n \in \mathbb{Z} : n \mod 2 \equiv 0 \mod 2\}\\ &= \{n \in \mathbb{Z} : n \equiv 0 \mod 2\}\\ &= \{n \in \mathbb{Z} : n \in 2\mathbb{Z}\}\\ &= 2\mathbb{Z}.\end{aligned}$$

Thus, the first isomorphisms theorem yields the previously discovered statement

$$\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2.$$

Date: September 29, 2008.

Proof. In previous lectures and homework we've already done the heavy lifting for this proof. In particular, we already know that $\mathrm{Ker}(\phi) \trianglelefteq G$ and $\mathrm{Im}(\phi) \leqslant G'$. The rest is now elementary.

For the sake of convenience, let $K = \mathrm{Ker}(\psi)$ and $I = \mathrm{Im}(\phi)$. We define a map

$$\psi : G/K \to I$$

by

$$\psi(Kg) = \phi(g)$$

for any $g \in G$. First, we note that ψ is a homomorphism.

$$\begin{aligned}\psi(Kx \cdot Ky) &= \psi(Kxy) \quad (\text{ by definition of } G/K) \\ &= \phi(xy) \\ &= \phi(x)\phi(y) \quad (\text{ because } \phi \text{ is a homomorphism }) \\ &= \psi(Kx)\psi(Ky).\end{aligned}$$

Also, ϕ is injective. Indeed, if $\psi(Kx) = \psi(Ky)$, then $\phi(x) = \phi(y)$, and hence

$$1 = \phi(x)\phi(y)^{-1} = \phi(x)\phi(y^{-1}) = \phi(xy^{-1}).$$

Therefore

$$xy^{-1} \in \mathrm{Ker}(\phi) = K.$$

Thus $Kxy^{-1} = K$ and therefore $Kx = Ky$. Therefore ψ is injective.

Finally, ψ is surjective. Let $a \in \mathrm{Im}(\phi)$. Then there is some $x \in G$ such that $a = \phi(x)$. But $Kx \in G/K$ for any $x \in G$. Then $\psi(Kx) = \phi(x) = a$. □

Definition 3. *Let* A *and* B *be groups. The* direct product *of* A *and* B *is a group which as a set is given by*

$$A \oplus B = \{(a, b) : a \in A, b \in B\},$$

and where multiplication is defined by

$$(a, b) * (a', b') = (aa', bb').$$

It's an easy and fun exercise to note that $A \oplus B$ is indeed a group.

Example 4. *Consider the map* $\phi : \mathbb{Z}_2 \oplus \mathbb{Z}_2 \to \mathbb{Z}_2$ *given by*

$$\phi(a, b) = a.$$

Then

$$\mathrm{Ker}(\phi) = \{(a, b) : a = 0\} = \{(0, 0), (0, 1)\}.$$

Note that ϕ *is onto. Therefore we have*

$$\mathbb{Z}_2 \oplus \mathbb{Z}_2 / \mathrm{Ker}(\phi) \cong \mathbb{Z}_2.$$

Definition 5. *The* centralizer $Z(G)$ *of a group* G *is the set of all elements that commute with all elements of* G*,*

$$Z(G) = \{x \in G : xg = gx \text{ for all elements } g \in G\}.$$

Proposition 6. *For any group* G*, the centralizer is a subgroup* $Z(G) \leqslant G$*.*

Proof. Note that $1 \in Z(G)$. Further if $x \in Z(G)$, then for $g \in G$

$$xg = gx \Rightarrow xgx^{-1} = g \Rightarrow gx^{-1} = x^{-1}g,$$

so $x^{-1} \in Z(G)$. Also, if $y \in Z(G)$, then

$$xyg = x(yg) = x(gy) = (xg)y = (gx)y = g(xy) = gxy.$$

So $Z(G)$ is closed, has inverses and the identity. □

Definition 7. *For elements* x *and* g *in a group* G*, the* conjugate *of* g *by* x *is* $xgx^{-1} \in G$*.*

Proposition 8. *The relation* $g \sim g'$ *if and only if* $g = xg'x^{-1}$ *for some* $x \in G$ *is an equivalence relation on the group* G*.*

Proof. Clearly $g \sim g$ by conjugating with the identity. Also, $g \sim g'$ implies there is some $x \in G$ such that

$$g = xg'x^{-1} \Rightarrow g' = x^{-1}gx,$$

and hence $g' \sim g$. Finally, $g \sim g'$ and $g' \sim g''$ imply $g = xgx^{-1}$ and $g' = yg''y^{-1}$ for some x and y in G. Thus

$$g = xg'x^{-1} = xyg''y^{-1}x^{-1} = (xy)g''(xy)^{-1},$$

and hence $g \sim g''$.

Definition 9. *The* conjugacy class *of an element* g *of a group* G *is the equivalence class of* g *in the above equivalence relation.*

Definition 10. *For an element* x *in the group* G*, the* inner automorphism *of* G *induced by* x *is the map*

$$T_x : G \to G$$

given by

$$T_x(g) = xgx^{-1}.$$

Remark 11. Note that T_x is indeed an automorphisms of G. It is elementary to check that T_x is a homomorphism, and its inverse is given by $T_{x^{-1}}$.

Proposition 12. *The set of all inner automorphisms of a group* G *form a group under composition denoted*

$$\mathrm{Inn}(G) = \{T_x : x \in G\}.$$

Proof. The proof is elementary and left to the reader.

Proposition 13. *For any group* G*, we have*

$$G/Z(G) \cong \mathrm{Inn}(G).$$

Proof. Consider the map $\psi : G \to \mathrm{Inn}(G)$ given by

$$x \mapsto T_x.$$

Note that for $g \in G$,

$$\psi(xy)(g) = T_{xy}(g) = (xy)g(xy)^{-1} = x(ygy^{-1})x^{-1} = T_x(T_y(g)) = (T_x \circ T_y)(g) = (\psi(x) \circ \psi(y))(g).$$

Hence ψ is a homomorphism. Further ψ is immediately seen to be surjective. Let's denote the identity map on G, which is the identity element of $\mathrm{Inn}(G)$, by I_G. Then

$$\begin{aligned}
\mathrm{Ker}(\psi) &= \{x \in G : \psi(x) = I_G\} \\
&= \{x \in G : T_x = I_G\} \\
&= \{x \in G : xgx^{-1} = g \text{ for all } g \in G\} \\
&= \{x \in G : xg = gx \text{ for all } g \in G\} \\
&= Z(G).
\end{aligned}$$

Therefore, by the first isomorphism theorem, we have

$$G/\,\mathrm{Ker}(\phi) \cong \mathrm{Im}(psi),$$

i.e.

$$G/Z(G) \cong \mathrm{Inn}(G).$$

Definition 14. *For a subset* $A \subset G$ *of a group* G*, the* normalizer of A in G *is*

$$N_G(A) = \{g \in G : gAg^{-1} = A\}.$$

Theorem 15 (Second Isomorphism Theorem)**.** *Let* G *be a group, and suppose* A *and* B *are subgroups such that* $A \leqslant N_G(B)$*. Then* $AB \leqslant G$, $B \trianglelefteq AB$, $A \cap B \trianglelefteq A$ *and*

$$AB/B \cong A/(A \cap B).$$

Proof. The proof is an application of the first isomorphisms theorem.

Theorem 16 (Third Isomorphism Theorem)**.** *Let* H *and* K *be normal subgroups of* G *and* $H \leqslant K$*. Then* $H \trianglelefteq K$, $K/H \trianglelefteq G/H$ *and*

$$(G/H)/(K/H) \cong G/K.$$