MATH 171 FALL 2008: CLASS 11

DAGAN KARP

ABSTRACT. In Class 11, we study examples of rings: polynomials, matrices and group rings.

1. POLYNOMIAL RINGS (AFTER P. GRILLET)

Intuitively, a polynomial in one indeterminate x and coefficients in a ring R is a linear combination

$$f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$$

of powers of x with coefficients $a_0, a_1, \ldots, a_n \in R$.

But what is x? What is a *variable* or an *indeterminate*? Note that x acts as a place holder, and that the polynomial is determined by its coefficients!

Definition 1. *A* polynomial *with one indeterminate and coefficients in a ring R is an infinite sequence*

$$a = (a_0, a_1, a_2, \ldots, a_n, \ldots)$$

of elements of R such that $a_n = 0$ for almost all n.

Remark 2. To say that $a_n = 0$ for almost all n is to say that there are only a finite number of n such that $a_n \neq 0$. In other words, the set

$$\{n \in \mathbb{N} \cup \{0\} : a_n \neq 0\}$$

is finite. Or equivalently, there exists some $N > 0$ such that $a_i = 0$ for all $i > N$.

We may define addition of polynomials componentwise,

$$(a_1, a_2, \ldots, a_n, \ldots) + (b_1, b_2, \ldots, b_n, \ldots) = (a_1 + b_1, a_2 + b_2, \ldots, a_n + b_n, \ldots).$$

In other words

$$a + b = c$$

where $c_n = a_n + b_n$.

Multiplication is defined by

$$ab = c \qquad \text{where} \qquad c_n = \sum_{i+j=n} a_i b_j.$$

Date: October 7, 2008.

1.1. **Exercise.** Let $a(x) = (x+1)^2$ be polynomial in one indeterminate and integer coefficients. Show that f is a polynomial according to the above definition. If $b(x) = x - 1$, compute ab according to the rules for multiplying polynomials. Does this agree with a direct computation of $a(x) \cdot b(x)$?

Proposition 3. *When* R *is a ring, polynomials with one indeterminate and coefficients in* R *form a ring, denoted* $R[x]$*. If* R *is commutative, then* $R[x]$ *is commutative.*

1.2. **Exercise.** Prove this proposition.

Definition 4. *The indeterminate* x *in* $R[x]$ *is defined by*

$$x = (0, 1, 0, 0, \dots, 0, \dots)$$

Now that x is defined, we can write polynomials in familiar form

$$(a_0, a_1, a_2, \dots, a_n, \dots) = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n,$$

if $a_i = 0$ for all $i > n$.

1.3. **Exercise.** In order to prove the above statement, calculate x^2 in $R[x]$, using the definition of the indeterminate x above and our knowledge of multiplication in $R[x]$. In general, what is x^n?

Definition 5. *The* degree *of a non-zero polynomial* $a(x) \in R[x]$ *is the largest* n *such that* $a_n \neq 0$. *Then* a_n *is the* leading coefficient *of* a *and* $a_n x^n$ *is the* leading term *of* a.

Proposition 6. *Let* R *be an integral domain, and let* $a(x), b(x)$ *be two non-zero polynomials in* $R[x]$.

(1) $\deg(ab) = \deg(a) + \deg(b)$
(2) *The units of* $R[x]$ *are the units of* R.
(3) $R[x]$ *is an integral domain.*

1.4. **Exercise.** Prove this proposition.

2. MATRIX RINGS

Definition 7. *For a ring* R *and* $n \in \mathbb{N}$*, let* $M_n(R)$ *denote the set of all* $n \times n$ *matrices with entries in* R*. For a matrix* $A \in M_n R$*, we denote by* $A_{i,j}$ *the entry of* A *in row* i *and column* j*.*

We may add and multiply matrices. Addition is defined componentwise,

$$A + B = C,$$

where

$$C_{i,j} = A_{i,j} + B_{i,j}.$$

Multiplication is the standard matrix multiplication:

$$(AB)_{i,j} = \sum_{k=1}^{n} A_{i,k} B_{k,j}.$$