MATH 171 FALL 2008: LECTURE 12

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ABSTRACT. The subjects of Class 12 are ring homomorphisms, subrings and quotients rings. This corresponds roughly with Section 7.3 of Dummit and Foote.

Definition 1. *A* subring S *of a ring* $(R, +, \cdot)$ *is a subgroup* $S \leqslant (R, +)$ *which is closed under the multiplicative structure of* R*.*

Example 2. $\mathbb{Q}$ *is a subring of* $\mathbb{R}$*.*

Proposition 3. *A subset* S *of the ring* R *is a subring if and only if* S *is closed under subtraction and multiplication.*

Proof. This follows immediately from the fact that a subset H of an Abelian group G is a subgroup if and only if H is closed under subtraction. □

Example 4. *The* center *of a ring* A *is the set of elements* $a \in A$ *such that* $ax = xa$ *for all* $x \in A$*. The center of* A *is a subring of* A*.*

Definition 5. *Let* R *and* S *be rings. A* ring homomorphism *is a map of sets* $\varphi : R \to S$ *such that, for all* $a, b \in R$*,*

(1)
$$\varphi(a + b) = \varphi(a) + \varphi(b)$$

(2)
$$\varphi(ab) = \varphi(a) \cdot \varphi(b)$$

Definition 6. *The* kernel *of the homomorphism* $\varphi : R \to S$ *is given by*

$$\mathrm{Ker}(\varphi) = \{r \in R : \varphi(r) = 0 \in S\}.$$

Definition 7. *An* isomorphism *of rings is a bijective homomorphism.*

Definition 8. *A subring* I *of* R *is a* left ideal *of* R *if* I *is closed under left multiplication by elements from* R*, i.e.* $rI \subset I$ *for all* $r \in R$*. Similarly,* I *is a* right ideal *of* R *if* I *is closed under right multiplication by elements of* R*, i.e.* $Ir \subset R$ *for all* $r \in R$*. A subring which is both a left and right ideal is called a* two sided ideal*, or simply* ideal*.*

Definition 9. *The* quotient ring R/I *of the ring* R *by the ideal* $I \subset R$ *is the quotient group of cosets* R/I *under the operations*

$$(r + I) + (s + I) = (r + s) + I \qquad (r + I) \cdot (s + I) = (r \cdot s) + I,$$

for all $r, s \in I$*.*

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Proposition 10. *For any ring* R *and ideal* I, R/I *is a ring.*

Theorem 11 (First isomporphism theorem for rings)**.** *If* $\varphi : R \to S$ *is a homomorphism of rings, then* $\operatorname{Ker}(\varphi)$ *is an ideal of* R, $\operatorname{Im}(\varphi)$ *is a subring of* S*, and*

$$R/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi).$$

Furthermore, if I *is any ideal of* R*, the map*

$$R \to R/I$$

defined by $r \mapsto r + I$ *is a surjective ring homomorphisms with kernel* I.

0.1. **Exercise.** In each group, one teammate prove each of the following.

(1) $\operatorname{Ker}(\varphi)$ is an ideal and $\operatorname{Im}\varphi$ is a subring.
(2) $R/\operatorname{Ker}\varphi \cong \operatorname{Im}\varphi$
(3) The map $R \to R/I$ is a surjective ring homomorphism.

Theorem 12 (Second isomorphism theorem for rings)**.** *Let* R *be a ring,* A *a subring and* B *an ideal of* R*. Then*

$$A + B = \{a + b : a \in A,\ b \in B\}$$

is a subring of R, $A \cap B$ *is an ideal of* A *and*

$$(A + B)/B \cong A/(A \cap B).$$

Theorem 13 (Third isomorphism theorem for rings)**.** *Let* I *and* J *be ideals of the ring* R *such that* $I \subset J$*. Then* J/I *is an ideal of* R/I *and*

$$(R/I)/(J/I) \cong R/J.$$

Theorem 14 (Fourth isomorphism theorem for rings)**.** *Let* I *be an ideal of* R*. The correspondence*

$$A \longleftrightarrow A/I$$

is an inclusion preserving bijection between the subrings A *of* R *containing* I *and the set of subrings of* R/I*. Further, a subring* A *containing* I *is an ideal of* R *if and only if* A/I *is an ideal of* R/I.

0.2. **Exercise.** Complete the following exercises as a group. Each of the following has one task for each member of the working group. They should be attacked separately, and then shared with the group to divide the labor and share ideas.

(1) (a) Show that $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic.
 (b) Show that $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic.
 (c) Is the map from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}/10\mathbb{Z}$ given by $x \mapsto 2x$ a ring homomorphism?

(2) Let $R = \mathbb{Z}[x]$, and I be all polynomials of degree at least 2, together with the zero polynomial

$$I = \{f \in \mathbb{Z}[x] : f = 0 \text{ or } \deg(f) \geqslant 2\}.$$

 (a) Show that I is a subring of R.
 (b) Show that I in an ideal of R.
 (c) Describe R/I.

(3) Let R be a commutative ring with 1, and $G = \{g_1, \dots, g_n\}$ be a finite group. The map $RG \to R$ given by

$$\sum_{i=1}^{n} a_i g_i \mapsto \sum_{i=1}^{n} a_i$$

is called the *augmentation map*.
 (a) Show that the augmentation map is a ring homomorphism
 (b) Describe the kernel of the augmentation map.
 (c) Show that the associated quotient ring is isomorphic to R.

(4) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, and let

$$H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

 (a) Show that $\mathbb{Z}[2]$ is a ring.
 (b) Show that H is a ring.
 (c) Show that H and $\mathbb{Z}[2]$ are isomorphic as rings.

(5) Let A and B be ideals of a ring R. The *product* of A and B is given by

$$AB = \{a_1 b_1 + \cdots + a_n b_n : a_i \in A,\ b_i \in B,\ n \in \mathbb{N}\}.$$

 (a) Show that AB is an ideal.
 (b) Show $AB \subset A \cap B$
 (c) If R is commutative with 1 and $A + B = R$, then show $A \cap B = AB$.

(6) (a) Determine all the ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

(b) Determine all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.

(c) Determine all ring homomorphisms from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$.