MATH 171 FALL 2008: LECTURE 13

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1. Properties of ideals

Throughout these notes, let R be a ring with 1.

Definition 1. *Let* $A \subset R$ *be a subset. Then* the (left) ideal generated by A *is the smallest (left) ideal of* R *containing* A, *and is denoted* (A).

Remark 2. Note that (A) is the intersection of all ideals I containing A.

$$(A) = \bigcap_{A \subset I} I$$

This intersection is nonempty as R is an ideal of itself containing A, and the intersection of nonempty ideals is an ideal.

Definition 3. *We define the notation* RA *by:*

$$RA = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n : r_i \in R,\ a_i \in A,\ n \in \mathbb{Z}\}.$$

Proposition 4. *If* R *is commutative, then* $(A) = RA$.

1.1. **Exercise.** Prove this proposition in the following steps.

(1) Show that RA is closed under addition and left multiplication by element of R.
(2) Show that RA contains A.
(3) Conclude that RA is an ideal containing A.
(4) Suppose that J is an ideal containing A. Show that $RA \subset J$.
(5) Conclude that RA is the left ideal generated by A.

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Remark 5. If R is commutative, then

$$AR = RA = (A).$$

Definition 6. *An ideal generated by a single element is called an* principal ideal*, and an ideal generated by a finite set is called a* finitely generated ideal*.*

1.2. **Exercise.** Consider the finitely generated ideal $(2, x)$ in $\mathbb{Z}[x]$.

(1) What do the elements of $(2, x)$ look like?
(2) Note that $(2, x)$ is a proper ideal, i.e. $(2, x) \neq \mathbb{Z}[x]$.
(3) Show that $(2, x)$ is not a principal ideal. Hint: Suppose otherwise, then $(2, x) = (a(x))$ for some $a(x) \in \mathbb{Z}[x]$.

Definition 7. *A proper ideal* M *of a ring* R *is* maximal *if whenever* I *is an ideal of* R *and* $M \subset I \subset R$*, then* $M = I$ *or* $I = R$*.*

Definition 8. *A proper ideal* P *of a commutative ring* R *is* prime *if* $ab \in P$ *implies* $a \in P$ *or* $b \in P$ *for any* $a, b in R$*.*

1.3. **Exercise.**

(1) Show that the ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is prime if and only if n is prime.
(2) Inspect the lattice of subgroups of $\mathbb{Z}/36\mathbb{Z}$ and show that (2) and (3) are maximal ideals.
(3) Show that the ideal $(x^2 + 1)$ is not prime in $\mathbb{Z}_2[x]$.