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MATH 131 NOTES

Recall 1. Last time we discussed Cauchy Sequences and Complete Metric Spaces.

Example 1. $\mathbb{R}$ is complete but $\mathbb{Q}$ is not.

Theorem 1. *Every metric space (X, d) has a completion $(X^\star, \Delta)$. In other words, $(X^\star, \Delta)$ is a complete metric space containing in X.*

Let $X^\star = \{\text{Cauchy sequences in } X\}/\sim$. We will say:

$$\{p_n\} \sim \{p'_n\} \iff \lim_{n\to\infty} d(p_n, p'_n) = 0.$$

Let $P, Q \in X^\star$. Then $P = [\{p_n\}]$, and $Q = [\{p'_n\}]$. We define:

$$\Delta(P, Q) = \lim_{n\to\infty} d(p_n, q_n).$$

We claim $(X^\star, \Delta)$ is a complete metric space and (X, d) is isometrically embedded in $(X^\star, \Delta)$. In other words, there is an injection $i : X \hookrightarrow X^\star$ such that $d(p, q) = \Delta(i(p), i(q))$.

Example 2. If $X = \mathbb{Q}$, then $X^\star = \mathbb{R}$. In particular, $X^\star$ is isometrically isomorphic to $\mathbb{R}$. In other there is a distance preserving bijection.

Remark 1. This is the other construction of $\mathbb{R}$. But Dedekind cuts are more hardcore.

Definition 1. A sequence $\{s_n\}$ in $\mathbb{R}$ is *monotonically increasing* if $s_n \le s_{n+1}$ for all n. Similarly $\{s_n\}$ is *monotonically decreasing* if $s_n \ge s_{n+1}$.

Theorem 2. *Let $\{s_n\}$ is monotonic (i.e., either). Then $\{s_n\}$ converges in $\mathbb{R}$ if and only if it is bounded.*

Proof. ($\Leftarrow$) Suppose $s_n \le s_{n+1}$ without loss of generality. Let $s = \sup(\text{range}(\{s_n\}))$. Then $s_n \le s$ for all n.

Let $\epsilon > 0$. Then $s - \epsilon < s$. Thus there exists $N \in \mathbb{N}$ such that $s - \epsilon < s_N \le s$ since s is a least-upper-bound by construction. Furthermore, since $s_{n+1} \ge s_n$ for all n. Thus if $n > N$ implies

$$s - \epsilon < s_N \le s_n \le s.$$

So $d(s_n, s) < \epsilon$. Therefore $s_n \to s$.

($\Rightarrow$) Already have shown that convergence sequences are bounded. ∎

Definition 2. Let $\{s_n\}$ be a sequence in $\mathbb{R}$. We define the *upper limit* of $\{s_n\}$ in $\mathbb{R} \cup \{\pm\infty\}$ (i.e. the completion of $\mathbb{R}$), is

$$\lim_{n\to\infty} \sup\{s_k | k > n\} = \limsup s_n.$$

Similarly the *lower limit* of s_n is:

$$\lim_{n\to\infty} \inf\{s_k | k > n\} = \liminf s_n.$$

Theorem 3. (Book definition) *Let $\{s_n\}$ be real. Let E be the set of all sub sequential limits of $\{s_n\}$. In other words:*

$$E = \{x \in \mathbb{R} \cup \{\pm\infty\} | s_{n_k} \to x \text{ for some subsequence } \{s_{n_k}\}\}.$$

Let $s^\star = \sup E$, $s_\star = \inf E$. Then $s^\star = \limsup s_n$ and $s_\star = \liminf s_n$.

Theorem 4. $s_n \to s \iff \limsup s_n = \liminf s_n = s$.

Theorem 5. *Let $\{s_n\}$ be real. Then*

(a) $\limsup s_n \in E$*, using the definition of E above.*

(b) If $x > \limsup s_n$, then $\exists N \in \mathbb{N}$ such that $n > N$ implies $s_n < x$. Moreover, $\limsup s_n$ *is the only number with this property. It is analogous for* $\liminf s_n$*.*

Proof. (Idea) Let $s_{n_k} \to t \in E$. Then $\liminf s_{n_k} = \limsup s_{n_k} = t$. But $\{s_{n_k}\} \subseteq \{s_n\}$. Notice:

$$\liminf s_n \leq \liminf s_{n_k} = t = \limsup s_{n_k} \leq \limsup s_n.$$

Thus

$$\liminf s_n \leq \inf E \leq \sup E \leq \limsup s_n.$$

But $\liminf s_n \in E$, $\liminf s_n \in E$. So $\liminf s_n = \inf E$, $\limsup s_n = \sup E$. ■

Example 3. Special Sequences. Let $p > 0$.

(a) $\lim_{n\to\infty} \frac{1}{n^p} = 0$.

(b) $\lim_{n\to\infty} p^{1/n} = 1$.

(c) $\lim_{n\to\infty} n^{1/n} = 1$.

(d) $\lim_{n\to\infty} \frac{n^a}{(1+p)^n} = 0$ for all $a \in \mathbb{R}$.

(e) If $|x| < 1$, then $\lim_{n\to\infty} x^n = 0$.