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MATH 131 NOTES

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Goal: Heine-Borel Theorem

Recall 1. If $F, k \subseteq X$ and if F is closed, and K is compact, then $F \cap K$ is compact.

Theorem 1. *(Nested closed intervals in $\mathbb{R}$ are nonempty.) Let $\dots \subseteq I_3 \subseteq I_2, \subseteq I_1$ be a sequence of nested closed intervals in $\mathbb{R}$. Let $I_n = [a_b, b_n]$. If $m > n$, then it follows by construction $a_n \leq a_m < b_m \leq b_n$. Then*

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Proof. Let $x = \sup\{a_i | i \in \mathbb{N}\}$. Note that any b_n is an upper-bound of $\{a_i\}$. Therefore $x \in \mathbb{R}$ exists. So $x \leq b_n$ for all n. Also note that $a_n \leq x$ for all n. Therefore $x \in I_n$ for all n. □

Remark 1. Same idea works for k-cells in $\mathbb{R}^k$. Note that a k-cell is of the form:

$$[a_1, b_1] \times [a_2, b_2] \times \cdots \times [a_k, b_k].$$

Fun Fact: Alternate proof that $\mathbb{R}$ is uncountable. Suppose $\mathbb{R} = \{x_1, x_2, x_3, \dots\}$. Let $I_n = \mathbb{R} \setminus \{x_1, x_2, \dots, x_n\}$. Then $\dots \subseteq I_3 \subseteq I_2 \subseteq I_1$. So $\{I_n\}$. If I_n is closed. Therefore:

$$\bigcap_{n=1}^{\infty} I_n \neq \emptyset.$$

Then there exists $z \in \bigcap I_n$, hence a contradiction.

Theorem 2. *Closed intervals in $\mathbb{R}$ are compact. (So are k-cells in $\mathbb{R}^k$.)*

Proof. Let $[a, b] \subseteq \mathbb{R}$. Suppose $[a, b]$ is not compact. Then there exists an open cover $\{G_\alpha\}$ with no finite subcover. Let $c_1 = \frac{a+b}{2}$. Then $\{G_\alpha\}$ covers $[a, c_1]$ and $[c, b]$. At least one has no finite subcover. Without loss of generality, it is $I_1 = [a, c_1]$. Let $c_2 = \frac{a+c_1}{2}$. Then $[a, c_2]$ of $[c_2, c_1]$ has no finite subcover. Call it I_2.

Repeat to obtain $\dots \subseteq I_3 \subseteq I_2 \subseteq I_1$. Notice these are nested closed intervals with no finite subcover of $\{G_\alpha\}$. Notice also that $|I_n| = 2|I_{n+1}|$. Then there exists a point $x \in \cap I_n$. But $x \in [a, b]$. Thus there exists an α such that $x \in G_\alpha$.

Notice there exists an $r > 0$ such that $B(x, r) \subseteq G_\alpha$. For n large enough, $I_n \subseteq B(x, r)$. But I_n has no finite subcover, so it can't be contained in any finite subcollection of $\{G_\alpha\}$. Hence a contradiction. □

Definition 1. The set $K \subseteq X$ is *bounded* if there exists $r > 0$ and $q \in X$, such that for all $p \in K$, $d(p, q) < r$.

Theorem 3. *Heine-Borel Theorem – In $\mathbb{R}$ (or $\mathbb{R}^k$), K is compact if and only if K is closed and bounded.*

Proof. Let $p \in K$. Then $K \subseteq \bigcap_{n \in \mathbb{N}} B(p,n)$, because this covers all of X. Assume K is compact. Then there exists a finite subcover. Therefore, $K \subseteq B(p,n_1) \cup \cdots \cup B(p,n_l)$. Thus $K \subseteq B(p,r)$ where $r = \max\{n_1,\ldots,n_l\}$. Thus K is bounded. Furthermore, we've already shown that if K is compact, then K is closed.

Conversely, suppose K is closed and bounded. Then there exists $r > 0$ such that $K \subseteq [-r,r]$. Furthermore, $[-r,r]$ is compact. Since K is closed subset of a compact set, K is compact. □

Corollary 1. *Let* $K \subseteq \mathbb{R}$. *If* K *is compact, then* $\sup k$ *exists and* $\sup k \in K$.

Proof. K is bound, hence $\sup k \in \mathbb{R}$. K is closed, hence $K' \subseteq K$. Furthermore, observe that $\sup K \in K'$ because it is a limit point. □

Example 1. Let $E \subseteq \mathbb{Q}$, $E = \{p \in \mathbb{Q} | 2 < p^2 < 3\}$. In $\mathbb{R}$, is E closed? Is it bounded?

Example 2. Let A be any set. Suppose A is infinite. Define

$$d(p,q) = \{0 \text{ if } p = q, 1 \text{ else}\}.$$

Is $A \subseteq A$ closed? Is A bounded. Yes on both accounts. Notice that $B(p, 1/2) = \{p\}$. Thus:

$$A = \bigcup_{p \in A} B(p, 1/2).$$

Notice that this cannot be reduced to a finite subcover. Hence, A is not compact.