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MATH 171 LECTURE 10

GROUP STRUCTURES

DEF: LET G BE A GP, $X \subseteq G$ A SUBSET. THE SUBGROUP OF G GENERATED BY X, DENOTED $\langle X \rangle$ IS THE SMALLEST SUBGROUP OF G CONTAINING X. THE ELEMENTS OF X ARE THE GENERATORS OF G.

EX: $X = \{2\}$, $G = \mathbb{Z}$.

$\langle 2 \rangle \leq G$. $\quad 2 \in \langle 2 \rangle \Rightarrow 2+2 \in \langle 2 \rangle$. $\quad 0 \in \langle 2 \rangle \leq G$.

$-2 \in \langle 2 \rangle \ldots \quad \langle 2 \rangle = 2\mathbb{Z}$.

PROP: LET $X \subseteq G$. $\langle X \rangle = \{ x_1^{n_1} x_2^{n_2} \cdots x_k^{n_k} \mid x_i \in X, n_i \in \mathbb{Z} \}$

(PRODUCTS OF ELEMENTS OF X AND THEIR INVERSES)

NOTE: SUPPOSE $G = \langle x_1, .., x_n \rangle$ AND $\varphi : G \to G'$ IS A HOM.

THEN φ IS DETERMINED BY ITS VALUE ON $x_1, .., x_n$.

$\left(a \in G \Rightarrow a = x_1^{n_1} \cdots x_k^{n_k} \Rightarrow \varphi(a) = \varphi\left(\prod x_i^{n_i}\right) = \prod \varphi(x_i)^{n_i}. \right)$

EX: $\varphi: \mathbb{Z} \to \mathbb{Z}$. $\mathbb{Z} = \langle 1 \rangle$.

$$\varphi(1) = a \Rightarrow \varphi(n) = \underbrace{\varphi(1) + \cdots + \varphi(1)}_{n \text{ TIMES}} = n\varphi(1) = n \cdot a$$

THE ISOMORPHISM THEOREMS

THM (FIRST ISO THM)

IF $\varphi: G \to H$ IS A HOM, THEN $\ker\varphi \trianglelefteq G$ AND $\varphi(G) \leq H$

$$G/\ker\varphi \cong \varphi(G).$$

NOTE: HOW FAR IS φ FROM AN ISO? MAY FAIL TO BE INJECTIVE OR SURJECTIVE. THIS RECTIFIES BOTH.

PROOF: DEFINE $\psi: G/\ker\varphi \to \operatorname{Im}(\varphi)$. LET $K = \ker\varphi$

BY $\psi(Kg) = \varphi(g)$.

- WELL DEFINED: SUPPOSE $Kg = Kg'$. THEN $g'g^{-1} \in \ker\varphi$, IE

$$\varphi(g'g^{-1}) = 1 \Rightarrow \varphi(g')\varphi(g)^{-1} = 1 \Rightarrow \varphi(g') = \varphi(g).$$

- HOM: $\psi(Kg\,Kg') = \psi(Kgg') = \varphi(gg') = \varphi(g)\varphi(g') = \psi(Kg)\psi(Kg')$.
- INJECTIVE: $\psi(Kg) = \psi(Kg') \Leftrightarrow \varphi(g) = \varphi(g') \Leftrightarrow \varphi(g'g^{-1}) = 1$

$$\Leftrightarrow g'g^{-1} \in K \Leftrightarrow Kg = Kg'.$$

• SURJECTIVE: $\varphi(g) \in \operatorname{Im}\varphi \Rightarrow Kg \in G/K$.

QED

EX: $\varphi: \mathbb{Z} \to \mathbb{Z}_2$, $a \mapsto a \bmod 2$.

THM: SECOND ISOMORPHISM THM:

LET G BE A GROUP, $A \leq G$, $N \trianglelefteq G$. THEN

$AN \leq G$, $N \trianglelefteq AN$, $A \cap N \trianglelefteq A$ AND

$$AN/N \cong A/(A \cap N).$$

PF: (GROUPWORK) $AN \leq G$, $N \trianglelefteq AN$, $A \cap N \trianglelefteq A$.

$$\varphi : A \to AN/N$$
$$a \mapsto Na.$$

• HOM: $\varphi(ab) = Nab = NaNb = \varphi(a)\varphi(b)$.

• SURJECTIVE: $AN = \{an \mid a \in A, n \in N\} = \bigcup_{a \in A} aN = \bigcup_{a \in A} Na = NA$.

$x \in AN/N \Rightarrow \exists a \in A, n \in N$ s.t. $x = Nan = anN = aN = Na$

THEN $\varphi(a) = Na = x$.

• $\ker\varphi$: $\varphi(a) = Na = N \Leftrightarrow a \in N$. $\ker\varphi = N$. BUT $a \in A$.

$\Rightarrow \quad A/\ker\varphi \cong AN/N$ IE $A/A \cap N \cong AN/N$.

THM (~~SECOND~~ THIRD ISO THM)

LET G BE A GP, $A \trianglelefteq G$, $B \trianglelefteq G$. IF $A \subseteq B$, THEN $A \trianglelefteq B$,

$B/A \trianglelefteq G/A$ AND

$$(G/A)/(B/A) \cong G/B.$$

PF. LET $\varphi : G/A \to G/B$ BE DEFINED BY

$$Ax \mapsto Bx.$$

WELL DEFINED: $Ax = Ay \Leftrightarrow xy^{-1} \in A \subseteq B \Rightarrow xy^{-1} \in B \Rightarrow xB = yB.$
$Bx = By.$

THUS φ IS WELL DEFINED.

HOM: $\varphi(Ax\,Ay) = \varphi(A\,xy) = Bxy = Bx\,By = \varphi(Ax)\,\varphi(Ay).$

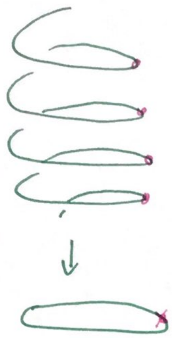
SURJ: $\forall\, Bx \in G/B,\ \varphi(Ax) = Bx.$

$\ker\varphi$: $\varphi(Ax) = Bx = B \Leftrightarrow x \in B$

SO $\ker\varphi = \{ Ax \in G/A \mid x \in B \} = B/A.$ QED

EX: $(\mathbb{Z}/6\mathbb{Z})/(3\mathbb{Z}/6\mathbb{Z}) \cong \mathbb{Z}/3\mathbb{Z}$

Ex: • $\mathbb{R}/\mathbb{Z} \cong S^1 = \mathbb{T} = \{e^{i\theta} \in \mathbb{C}\}$



$$\varphi : \mathbb{R} \to \mathbb{T}$$
$$a \mapsto e^{2\pi i a}$$

φ IS A SURJ HOM W/ $\ker \varphi = \mathbb{Z}$.

• LET V BE A FINITE DIM'L VECTOR SPACE OVER $\mathbb{R}$.

$SL(V) = \{T : V \to V \mid T \text{ IS LINEAR OF DET } 1\}$

$GL(V) = \{T : V \to V \mid T \text{ IS INVERTIBLE}\}$

$SL(V) \trianglelefteq GL(V)$, $\quad GL(V)/SL(V) \cong \mathbb{R}^\times$.

• CAYLEY'S THM: EVERY GROUP ISOMORPHIC TO A SUBGROUP OF A PERMUTATION GROUP $S_X = \{f : X \to X \mid f \text{ IS A BIJECTION}\}$

$$\varphi : G \to S_G$$
$$a \mapsto \lambda_a : G \to G$$
$$\lambda_a(x) = ax.$$

• $\lambda_a \in S_G$: $\lambda_a(x) = \lambda_a(y) \Leftrightarrow ax = ay \Leftrightarrow x = y$
$\lambda_a(a^{-1}x) = x$ SO λ_a SURJECTIVE. $\Big\}\ \lambda_{a^{-1}} = (\lambda_a)^{-1}$.

• $\varphi(ab) = \lambda_{ab}$. $\lambda_{ab}(x) = abx = a(bx) = \lambda_a(\lambda_b(x)) = \lambda_a \circ \lambda_b(x)$

$\varphi(ab) = \varphi(a)\varphi(b)$

$\circ \ker \varphi = \{ a \in G \mid \lambda_a = I_G \}$

$= \{ a \in G \mid ax = x \ \ \forall x \in G \}$

$= \{ 1 \in G \}.$

THUS $G \cong \operatorname{Im}(\varphi) \leq S_G$. QED

<u>DEF:</u> LET G_1, G_2 BE GPS. THE DIRECT PRODUCT OF G_1 AND G_2 IS

$$G_1 \times G_2 = \{ (g_1, g_2) \mid g_1 \in G_1, g_2 \in G_2 \}.$$

- $(g_1, g_2) \cdot (g_1', g_2') = (g_1 g_1', g_2 g_2')$
- $(e_1, e_2) \cdot (g_1, g_2) =$

<u>EX:</u> $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong \mathbb{Z}_4$?