MATH 171: LECTURE 11

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DIRECT PRODUCTS AND THE FTFGAG

DEF: LET $(G_1, *)$, $(G_2, \star)$ BE GPS. THE **DIRECT PRODUCT** OF G_1 AND G_2 IS

$$G_1 \times G_2 = \{ (g_1, g_2) \mid g_1 \in G_1 ,\ g_2 \in G_2 \}$$

$$(g_1, g_2) \cdot (g_1', g_2') = (g_1 * g_1',\ g_2 \star g_2')$$

DEF: THE DIRECT PRODUCT OF $G_1, G_2, G_3, \ldots$ IS $\prod_{i=1}^{\infty} G_i = G_1 \times G_2 \times G_3 \times \cdots$

WITH $(g_1, g_2, g_3, \ldots) \cdot (h_1, h_2, h_3, \ldots) = (g_1 h_1, g_2 h_2, \ldots)$

EX: $\mathbb{R} \times \mathbb{R} \cong \mathbb{R}^2$ (WITH VECTOR ADDITION)

$$(a,b) + (c,d) = (a+c, b+d)$$

EX: $G_1 = \mathbb{Z}$, $G_2 = S_3$, $G_3 = GL_2(\mathbb{R})$

$$\left(n, \sigma, \begin{pmatrix} a & b \\ c & d \end{pmatrix}\right) \cdot \left(m, \tau, \begin{pmatrix} e & f \\ g & h \end{pmatrix}\right)$$

$$= \left(n+m, \sigma \circ \tau, \begin{pmatrix} ae+bg & af+bh \\ ce+dg & cf+dh \end{pmatrix}\right)$$

THM: THE DIRECT PRODUCT OF THE GROUPS $G_1, G_2, \dots, G_n$ IS A GROUP OF ORDER $|G_1| \cdot |G_2| \cdots |G_n|$.

PF: (GROUPWORK?)

PROP: LET $G_1, \dots, G_n$ BE GROUPS. $G = G_1 \times \dots \times G_n$

(1) LET $\iota_i : G_i \to G$ BE GIVEN BY

$\iota_i(g_i) = (1, 1, \dots, g_i, \dots, 1)$. iTH INJECTION MAP.

iTH POSITION

$$G_i \cong \{(1, 1, \dots, g_i, 1, \dots, 1) \mid g_i \in G_i\} = \mathrm{Im}(\iota_i)$$

THEN $G/G_i \cong G_1 \times G_2 \times \dots \times G_{i-1} \times G_{i+1} \times \dots \times G_n$

(2) LET $\pi_i : G \to G_i$ BE GIVEN BY

$$\pi_i(g_1, \ldots, g_n) = g_i$$

(i^{TH} PROJECTION MAP)

THEN π_i IS A SURJECTIVE HOM AND

$$\ker \pi_i = \{(g_1, g_2, \ldots, g_{i-1}, 1, g_{i+1}, \ldots, g_n) \mid g_j \in G_j, \; j \neq i\}$$

$$\cong G_1 \times G_2 \times \cdots \times G_{i-1} \times G_{i+1} \times \cdots \times G_n.$$

DEF: A GROUP G IS THE INTERNAL DIRECT SUM OF SUBGROUPS $A_1, A_2, \ldots, A_n$ IN CASE

(a) $a_i a_j = a_j a_i \quad \forall a_i \in A_i, a_j \in A_j, \; i \neq j$

(b) EVERY $x \in G$ CAN BE WRITTEN UNIQUELY AS $x = a_1 a_2 a_3 \cdots a_n$ WHERE $a_i \in A_i$ ALL i.

THM: (1) $A_1 \oplus \cdots \oplus A_n \cong A_1 \times A_2 \times \cdots \times A_n$

(2) $G_1 \times G_2 \times \cdots \times G_n \cong \mathrm{Im}(\iota_1) \oplus \mathrm{Im}(\iota_2) \oplus \cdots \oplus \mathrm{Im}(\iota_n)$

THM: $G \cong A_1 \oplus \cdots \oplus A_n$ IFF

(I) $A_i \trianglelefteq G \quad \forall i$

(II) $(A_1 A_2 \cdots A_i) \cap A_{i+1} = 1 \quad \forall i < n$

(III) $G = A_1 A_2 \cdots A_n = \{a_1 a_2 \cdots a_n \mid a_i \in A_i\}$

EX: DEFINE V_4 BY

	1	a	b	c
1	1	a	b	c
a	a	1	c	b
b	b	c	1	a
c	c	b	a	1

THEN $V_4 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2$

(GROUPWORK)

THM: (THE FUNDAMENTAL THEOREM OF FINITELY GENERATED ABELIAN GROUPS)

LET G BE A FINITELY GENERATED ABELIAN GROUP. THEN

☺ $$G \cong \mathbb{Z}^r \times \mathbb{Z}_{n_1} \times \mathbb{Z}_{n_2} \times \cdots \times \mathbb{Z}_{n_k}$$

FOR INTEGERS $r, n_1, \ldots, n_k$ SUCH THAT

(a) $r \geq 0$, $n_i \geq 2$ FOR ALL i

(b) $n_i \mid n_{i+1}$ (n_1 DIVIDES n_2, ...)

ALSO, THE EXPRESSION ☺ IS UNIQUE.

IF $G \cong \mathbb{Z}^t \times \mathbb{Z}_{m_1} \times \cdots \times \mathbb{Z}_{m_l}$

SATISFYING (a), (b), THEN $r = t$, $k = l$,

AND $m_i = n_i$ FOR ALL i.

COR: IF n IS THE PRODUCT OF DISTINCT PRIMES, THEN THE ONLY ABELIAN GROUP OF ORDER n IS $\mathbb{Z}_n$.

NOTE: $\mathbb{Z}_{ab} \cong \mathbb{Z}_a \times \mathbb{Z}_b$ IFF $(a,b)=1$.

(CHINESE REMAINDER THM)

EX: $|G|=4$. $G \cong \mathbb{Z}_4$ OR $\mathbb{Z}_2 \oplus \mathbb{Z}_2$

$|G|=6$: $G \cong \mathbb{Z}_6 \cong \mathbb{Z}_2 \oplus \mathbb{Z}_3$

$|G|=180$ $G \cong \mathbb{Z}_{180}$ OR | $2^2 \cdot 3^2 \cdot 5$

(GROUP WORK)

$\mathbb{Z}_2 \times \mathbb{Z}_{90}$ OR | $2,\ 2 \cdot 3^2 \cdot 5$

$\mathbb{Z}_3 \times \mathbb{Z}_{60}$ OR | $3,\ 2^2 \cdot 3 \cdot 5$

$\mathbb{Z}_6 \times \mathbb{Z}_{30}$ | $2 \cdot 3,\ 2 \cdot 3 \cdot 5$