MATH 171 LECTURE 5

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S_n AND D_n

<u>DEF:</u> LET A, B BE SETS AND

$$f : A \to B$$

BE A MAP OF SETS.

(0) $f \subseteq A \times B$ SUCH THAT $\forall a \in A\ \exists!\ b \in B$ WITH $(a,b) \in f$. WRITE $f(a) = b$.

(1) THE DOMAIN OF f IS A

(2) THE CODOMAIN OF f IS B

(3) THE RANGE OR IMAGE OF f IS

$$f(A) = \{ f(a) \mid a \in A \} = \{ b \in B \mid b = f(a) \text{ FOR SOME } a \in A \}$$

(4) f IS ONTO OR SURJECTIVE IF

RANGE $(f) =$ CODOMAIN (f)

(5) f IS ONE TO ONE OR INJECTIVE

$f(a_1) = f(a_2) \Rightarrow a_1 = a_2.$

(6) f IS BIJECTIVE IF IT IS

INJECTIVE AND SURJECTIVE.

EX: $f: \mathbb{Z} \to \mathbb{Z} \quad f(x) = 2x.$

DOM $(f) = \mathbb{Z}$ CODOM $(f) = \mathbb{Z}$ RANGE $(f) = 2\mathbb{Z}$

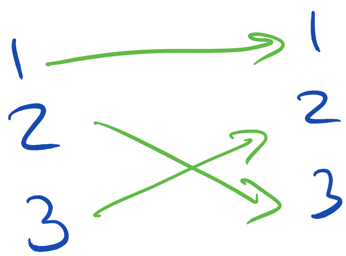
$f(x) = f(y) \Leftrightarrow 2x = 2y \Leftrightarrow x = y.$

f IS INJECTIVE AND NOT SURJECTIVE.

DEF: A BIJECTION OF A FINITE SET

TO ITSELF IS ALSO CALLED A PERMUTATION.

EX:



PERMUTATION!

NOT A PERM.
NOT BIJECTIVE
NOT SURJ.
NOT INJ.

DEF: S_n IS THE SET OF ALL PERMUTATIONS OF THE SET $\{1, 2, \ldots, n\}$.

THM: $(S_n, \circ)$ IS A GROUP UNDER COMPOSITION OF FUNCTIONS. IT HAS $n!$ ELEMENTS.

PF: LET $I : \{1, 2, \ldots, n\} \to \{1, 2, \ldots, n\}$

DENOTE THE IDENTITY MAP

$$I(k) = k \quad \forall k = 1, \ldots, n.$$

LET $\sigma \in S_n$. THEN

$$I \circ \sigma(k) = \sigma(k)$$
$$\sigma \circ I(k) = \sigma(k)$$

THUS I IS THE IDENTITY IN $(S_n, \circ)$

LET $\sigma \in S_n$. FOR $b \in \{1, \ldots, n\}$

$\exists\, a \in \{1, \ldots, n\}$ s.t. $\sigma(a) = b$.

DEFINE $\tau \in S_n$ BY

$\tau(b) = a$. THEN

$$\tau \circ \sigma(a) = \tau(b) = a.$$
$$\sigma \circ \tau(b) = \sigma(a) = b.$$

THUS $\tau \circ \sigma = I = \sigma \circ \tau$.

HENCE $\tau = \sigma^{-1} \in S_n$.

FINALLY, IF $\sigma, \tau \in S_n$, THEN $\sigma \circ \tau \in S_n$

AS THE COMPOSITION OF BIJECTIONS IS

BIJECTIVE. ALSO, COMPOSITION OF

FUNCTIONS IS ASSOCIATIVE. QED

DEF: A PERMUTATION $\sigma \in S_n$ IS

A CYCLE OF LENGTH k OR

A k-CYCLE IF $\exists$ $a_1, \dots, a_k \in \{1, \dots, n\}$

SUCH THAT

$$\begin{aligned} \sigma(a_1) &= a_2 \\ \sigma(a_2) &= a_3 \\ &\vdots \\ \sigma(a_{k-1}) &= \sigma(a_k) \end{aligned} \qquad \sigma(a_k) = a_1$$

NOTATION: WE WRITE

$$\sigma = (a_1 a_2 \ldots a_n)$$

EX: $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 3 & 2 \end{pmatrix} = (1)(23)$

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 2 & 3 & 1 & 5 & 4 \end{pmatrix} = (123)(45)$$

DEF: TWO CYCLES σ AND τ ARE DISJOINT IF $\sigma = (a_1, \ldots, a_k)$ $\tau = (b_1, \ldots, b_\ell)$ AND $a_i \neq b_j$ FOR ANY i, j.

THM: EVERY PERMUTATION CAN BE WRITTEN AS A PRODUCT OF DISJOINT CYCLES.

THM: DISJOINT CYCLES PERMUTE.

IE IF σ AND τ ARE DISJOINT CYCLES IN S_n, THEN $\sigma\tau = \tau\sigma$.

NOTE:

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 4 & 5 \end{pmatrix} = (23)$$ OMIT FIXED TERMS

$$\begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 2 & 5 & 4 \end{pmatrix} = (23)(45) = (45)(23)$$

DEF: A TRANSPOSITION IS A CYCLE OF LENGTH 2.

(ODD/EVEN. ORDER OF PERM = GCD.)

THM: EVERY PERMUTATION CAN BE WRITTEN AS A PRODUCT OF TRANSPOSITIONS

EX: $(132) \in S_3$.

$(132) = (12)(13)$

PF: $(a_1 a_2 \ldots a_k) = (a_1 a_k)(a_1 a_{k-1}) \cdots (a_1 a_2)$

SO EVERY CYCLE IS THE PRODUCT OF TRANSPOSITIONS, AND EVERY PERMUTATION IS A PRODUCT OF CYCLES.

LEMMA: IF $I = \beta_1 \beta_2 \ldots \beta_r$ WHERE β_i ARE TRANSPOSITIONS, THEN r IS EVEN.

PF: $r \neq 1$ (B/C I NOT A 2-CYCLE)

IF $r=2$ WE ARE DONE. NOW USE INDUCTION ON r.

THM (ALWAYS EVEN OR ALWAYS ODD)

LET $\alpha \in S_n$, $(n>1)$. SUPPOSE

$$\alpha = \beta_1 \beta_2 \cdots \beta_r \quad \text{AND} \quad \alpha = \gamma_1 \gamma_2 \cdots \gamma_s$$

WHERE β_i AND γ_j ARE 2-CYCLES.

THEN r AND s ARE BOTH EVEN OR BOTH ODD.

PF: BY ASSUMPTION

$$\beta_1 \beta_2 \cdots \beta_r = \gamma_1 \cdots \gamma_s .$$

THUS

$$I = \gamma_1 \cdots \gamma_s \beta_r^{-1} \beta_{r-1}^{-1} \cdots \beta_1^{-1}$$

$$= \gamma_1 \cdots \gamma_s \beta_r \beta_{r-1} \cdots \beta_1$$

HENCE, BY THE LEMMA, $r+s$ IS EVEN.

THEREFORE r AND s ARE BOTH EVEN OR BOTH ODD.

DEF: LET $\alpha = \beta_1 \cdots \beta_r$, β_i 2-CYCLE.
IF r IS EVEN WE SAY α IS AN
EVEN PERMUTATION. IF r IS ODD WE SAY
α IS AN ODD PERMUTATION.

THM: THE SET OF EVEN PERMUTATIONS
IN S_n FORMS A SUBGROUP OF S_n.

DEF: THE SUBGROUP OF EVEN PERMUTATIONS
OF S_n IS CALLED THE ALTERNATING
GROUP OF DEGREE n, DENOTED A_n.

NOTE: $|A_n| = n!/2$.

FROM CONTEMPORARY ABSTRACT ALGEBRA BY JOSEPH GALLIAN

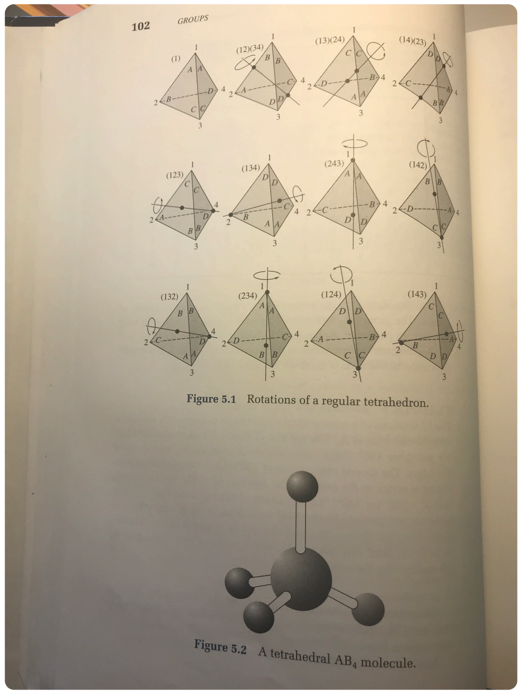

102 GROUPS

Figure 5.1 Rotations of a regular tetrahedron.

Figure 5.2 A tetrahedral AB_4 molecule.

EX: A_4 CORRESPONDS TO THE 12 ROTATIONAL SYMMETRIES OF THE TETRAHEDRONS

AB_4 MOLECULE EXAMPLE: METHANE CH_4.