MATH 171 : LECTURE 7

DAGAN KARP

COSETS + LAGRANGE'S THM

RECALL: A PARTITION $\overline{P}$ OF A SUBSET S IS A COLLECTION OF SUBSETS $P=\{E_\alpha\}_{\alpha\in I}$, $E_\alpha \subseteq S$, SUCH THAT

(1) $\bigcup_{\alpha\in I} E_\alpha = S$

(2) $E_\alpha \neq E_\beta \Rightarrow E_\alpha \cap E_\beta = \emptyset$

PROP: LET $\sim$ BE AN EQ REL ON S. THEN $P=\{[a]\}_{a\in S}$ IS A PARTITION OF S.

PF: (1) LET $a\in S$. THEN $a\sim a$ AND SO $a\in[a]$. THUS $a\in \bigcup_{a\in S}[a]$. THUS $S\subseteq \bigcup_{a\in S}[a]$

LET $x \in \bigcup_{a \in S'} [a]$. THEN $\exists\, a_0 \in S'$

SUCH THAT $x \in [a_0]$. THUS $x \sim a_0$ AND $x \in S'$.

THEREFORE $\bigcup_{a \in S'} [a] \subseteq S'$. THUS $S' = \bigcup_{a \in S'} [a]$.

(2) WE PROVE THE CONTRAPOSITIVE.

$$[a] \cap [b] \neq \emptyset \Rightarrow [a] = [b].$$

SUPPOSE $c \in [a] \cap [b]$. THEN $c \in [a]$ AND $c \in [b]$

THUS $c \sim a$ AND $c \sim b$, IE $a \sim c$ AND $c \sim b$. THUS

$a \sim b$. THEREFORE $[a] = [b]$. QED

COROLLARY. ANY TWO RIGHT (OR LEFT) COSETS OF

$H \leq G$ ARE IDENTICAL OR DISJOINT.

LEMMA: LET $H \leq G$, $x, y \in G$. THEN

$$|Hx| = |Hy|.$$

PROOF: WE CONSTRUCT A BIJECTION

$$\varphi : Hx \to Hy$$

DEFINED BY $\varphi(hx) = hy$. $\left(\varphi(a) = ax^{-1}y\right)$

SURJECTIVE: $\forall b \in Hy \; \exists h \in H$ s.t. $b = hy$.

THUS $\exists \; a \in Hx$ s.t. $a = hx$. THUS $\varphi(a) = b$.

INJECTIVE: LET $a_1, a_2 \in Hx$ AND SUPPOSE

$$\varphi(a_1) = \varphi(a_2).$$

$a_1, a_2 \in Hx \Rightarrow \exists \; h_1, h_2 \in H$ st $a_i = h_i x$.

THUS $\varphi(a_1) = \varphi(a_2) \Leftrightarrow h_1 y = h_2 y$

$\Rightarrow h_1 = h_2 \Rightarrow h_1 x = h_2 x \Rightarrow a_1 = a_2$.

QED

COROLLARY! LET H BE A FINITE SUBGROUP OF G.

FOR ANY $x \in G$, $|Hx| = |H|$.

PF: $|Hx| = |He| = |H|$.

LAGRANGE'S THM: (1736-1813) (NAPOLEON, METRIC SYSTEM)

LET H BE A SUBGROUP OF A FINITE GROUP G. THEN $|H|$ IS A DIVISOR OF $|G|$.

PROOF: WE PARTITION G INTO DISTINCT COSETS OF H.

FOR EACH $x \in G$, THERE IS A UNIQUE COSET Hx SUCH THAT $x \in Hx$. (NOTE; IF $x \in Hy$ THEN $Hy = Hx$) THUS, AS ABOVE,

$$\bigcup_{x \in G} Hx = G$$

BUT $|G|$ IS FINITE. THUS THERE ARE FINITELY MANY DISTINCT COSETS Hx. CALL THIS NUMBER h.
SO $\exists\ x_1, .., x_h$ SUCH THAT

$$Hx_1 \cup Hx_2 \cup \dots \cup Hx_h = G$$

AND $$Hx_i \cap Hx_j = \emptyset \quad \text{IF } i \neq j.$$

THUS $|G| = |Hx_1| + \dots + |Hx_h|$.

BUT $|Hx_1| = |Hx_2| = \dots = |Hx_h| = |H|$.

THUS $|G| = \sum_{i=1}^{h} |Hx_i| = h|H|$.

HENCE $|G|/|H| = h$. QED

<u>DEF:</u> THE <u>INDEX</u> OF $H \leq G$ IS THE NUMBER OF DISTINCT RIGHT COSETS OF H IN G. IT IS DENOTED $[G:H]$.

NOTE: IF $|G| < \infty$, $H \leq G$, THEN

$$|G| = |H|\,[G:H]$$

COROLLARY: LET G BE A FINITE GROUP AND $a \in G$. THEN $|a| \mid |G|$.

PF: $|a| = |\langle a \rangle|$ AND $\langle a \rangle \leq G$. THUS $|\langle a \rangle| \mid |G|$ SO $|a| \mid |G|$.

EX: THE ORDER OF EVERY ELEMENT IN $\mathbb{Z}_n$ DIVIDES n. SO IF p IS PRIME, $a \in \mathbb{Z}_p$, $a \neq 1 \Rightarrow |a| = p \Rightarrow \langle a \rangle = \mathbb{Z}_p$.

COROLLARY: LET G BE A FINITE GROUP AND $a \in G$. THEN

$$a^{|G|} = 1.$$

PF: $|a| \mid |G|$. THUS $\exists\, k$ SUCH THAT $|G| = k|a|$. THUS

$$a^{|G|} = a^{k|a|} = \left(a^{|a|}\right)^{k} = 1^{k} = 1. \quad \text{QED}$$

PROPOSITION: THE NUMBER OF (DISTINCT) RIGHT COSETS OF A SUBGROUP IS EQUAL TO THE NUMBER OF ITS (DISTINCT) LEFT COSETS.

PF:

$$Hx \longrightarrow x^{-1}H$$

$$Hy^{-1} \longleftarrow yH$$

BIJECTION