MATH 171 LECTURE 9

DAGAN KARP

NORMAL SUBGROUPS

DEF: A SUBGROUP N OF G IS **NORMAL** IF

$$aNa^{-1} \subseteq N$$

FOR ALL $a \in G$. DENOTED $N \trianglelefteq G$.

NOTE: $aNa^{-1} = \{ana^{-1} \mid n \in N\}$.

THUS

$$N \trianglelefteq G \iff \text{FOR ALL } a \in G, n \in N,$$

$$ana^{-1} \in N.$$

LEMMA: LET $N \leq G$. TFAE

(1) $N \trianglelefteq G$.

(2) $\forall a \in G \quad aNa^{-1} = G$

(3) $\forall a \in G \quad aN = Na$ (RIGHT & LEFT COSETS EQUAL)

(4) $\forall a, b \in G \quad NaNb = Nab$

NOTE: $NaNb = \{n_1 a n_2 b \mid n_1, n_2 \in N\}$

PF: (1) $\Rightarrow$ (2) LET $a \in G$. THEN $a^{-1} \in G$.

THUS $a^{-1}Na \subseteq N$. SO $\forall n \in N$, $a^{-1}na \in N$.

THUS $a(a^{-1}na)a^{-1} \in aNa^{-1}$. SO $n \in aNa^{-1}$

FOR ALL $n \in N$, ie $N \subseteq aNa^{-1}$. BUT $aNa^{-1} \subseteq N$

BY ASSUMPTION. THUS $aNa^{-1} = N$.

(2) $\Leftrightarrow$ (3) $aNa^{-1} = N \Rightarrow aN = Na$

$\forall n \in N \; \exists n' \in N$ s.t. $ana^{-1} = n'$. THUS

$an = n'a$. THUS $aN \subseteq Na$. SIMILARLY $Na \subseteq aN$.

(3) $\Rightarrow$ (4) $NaNb = N(aN)b = N(Na)b$

$= NNab = Nab$.

(4) $\Rightarrow$ (1) $NaNa^{-1} = Naa^{-1} = N$.

LET $ana^{-1} \in aNa^{-1}$.

THEN FOR $n' \in N$, $n'ana^{-1} = n''$.

So $ana^{-1} = n''(n')^{-1} \in N$.

THUS $aNa^{-1} \subseteq N$. QED

EX: • IF G IS ABELIAN, THEN $H \leq G \Rightarrow H \trianglelefteq G$

(EVERY SUBGROUP OF AN ABELIAN SUBGROUP IS NORMAL)

• $Z(G) \trianglelefteq G$, $Z(G) = \{h \in G \mid ah = ha \ \forall a \in G\}$

(THE CENTER OF G IS NORMAL IN G)

• LET $G = S_3$ $H = \langle (13) \rangle$. THEN

$H \leq G$ BUT $H \ntrianglelefteq G$.

(NOT ALL SUBGROUPS ARE NORMAL)

LET'S VERIFY. $|G| = |H| [G:H]$ (LAGRANGE)

THUS $[G:H] = 6/2 = 3$. SO $\exists$ 3

DISTINCT RIGHT AND LEFT COSETS. ARE

THEY EQUAL? DOES $gH = Hg \ \forall g \in G$?

$G = \{(1), (12), (13), (23), (123), (132)\}$ $H = \{(1), (13)\}$

$He = \{he \mid h \in H\} = H$

$H(123) = \{(123), (13)(123)\} = \{(123), (12)\}$

$\shortparallel$

$H(12) = \{(12), (13)(12)\} = \{(12), (123)\}$

$H(23) = \{(23), (13)(23)\} = \{(23), (132)\}$

(THESE MUST BE ALL 3 DISTINCT RIGHT COSETS)

$(12)H = \{(12), (12)(13)\} = \{(12), (132)\}$

THUS $(12)H \neq H(12)$. THEREFORE

$H \ntrianglelefteq G$.

DEF: LET $\varphi : G \to G'$ BE A HOM.

THE KERNEL OF φ IS

$$\ker(\varphi) = \{x \in G \mid \varphi(x) = 1\}$$

PROP: φ IS INJECTIVE $\Leftrightarrow$ $\ker\varphi = \{1\}$.

PF: φ NOT INJECTIVE $\Leftrightarrow \exists\, x \neq y \in G$ s.t.

$\varphi(x) = \varphi(y) \Leftrightarrow \varphi(x)\varphi(y)^{-1} = 1$

$\Leftrightarrow \varphi(xy^{-1}) = 1 \Rightarrow \ker\varphi \neq \{1\}$.

$\ker\varphi \neq \{1\} \Rightarrow \exists\, x \neq 1$ s.t. $\varphi(x) = 1$.

THEN $\varphi(x) = \varphi(1) = 1$ BUT $x \neq 1$. THUS

φ NOT INJECTIVE.

THM: LET $\varphi: G \to G'$ BE A HOM.

THEN $\ker\varphi \trianglelefteq G$.

PF: LET $a \in G$, $x \in \ker(\varphi)$. IE $\varphi(x) = 1$.

THEN $\varphi(axa^{-1}) = \varphi(a)\varphi(x)\varphi(a^{-1}) = \varphi(a)\varphi(a^{-1})$

$= \varphi(aa^{-1}) = \varphi(1) = 1$

THUS $axa^{-1} \in \ker\varphi$ FOR ALL $x \in \ker\varphi$, $a \in G$.

THUS $a \ker\varphi\, a^{-1} \subseteq \ker\varphi$. THUS $\ker\varphi \trianglelefteq G$.

THEOREM: LET $N \trianglelefteq G$. THEN THE SET OF RIGHT COSETS $\{Na \mid a \in G\}$ IS A GROUP UNDER THE OPERATION

$$Na \cdot Nb = Nab.$$

IT IS CALLED THE QUOTIENT GROUP OF G BY N AND DENOTED G/N.

PF: • CLOSURE: LET $Na, Nb \in G/N$.

THEN $\underline{Na \cdot Nb = Nab} \in G/N$. (b/c $N \trianglelefteq G$)

• ASSOCIATIVE b/c G IS ASSOCIATIVE

$$Na(Nb\,Nc) = Na(N_{bc}) = N_{a(bc)} = N_{(ab)c} = (NaNb)Nc$$

- IDENTITY: $\forall\, Na \in G/N$

$$NeNa = Na = NaNe.$$

THUS $N = Ne$ IS THE IDENTITY IN G/N

- INVERSES: $Na^{-1}Na = NaNa^{-1} = N_e = N$. QED

EX: $\mathbb{Z}/2\mathbb{Z} = \{2\mathbb{Z} + n \mid n \in \mathbb{N}\}$

$$= \{2\mathbb{Z},\ 2\mathbb{Z}+1\}$$

$2\mathbb{Z} + 2\mathbb{Z} = 2\mathbb{Z}$ $\qquad$ $2\mathbb{Z} + 2\mathbb{Z}+1 = 2\mathbb{Z}+1$

$2\mathbb{Z}+1 + 2\mathbb{Z}+1 = 2\mathbb{Z}$

+	$2\mathbb{Z}$	$2\mathbb{Z}+1$
$2\mathbb{Z}$	$2\mathbb{Z}$	$2\mathbb{Z}+1$
$2\mathbb{Z}+1$	$2\mathbb{Z}+1$	$2\mathbb{Z}$

$+_2$	0	1
0	0	1
1	1	0

THUS $\mathbb{Z}/2\mathbb{Z} \cong \mathbb{Z}_2$

$$\varphi(2\mathbb{Z}) = 0$$
$$\varphi(2\mathbb{Z}+1) = 1$$

(THIS IS MODULAR ARITHMETIC)

DEF: LET $N \trianglelefteq G$. THE PROJECTION HOMOMORPHISM IS DEFINED BY

$$\pi : G \to G/N$$
$$a \mapsto Na.$$

NOTE: $\pi(ab) = Nab = NaNb = \pi(a)\pi(b)$

REMARK: $\ker(\pi) = N$.

$$\pi(a) = N \iff a \in N.$$

(SO EVERY $\ker \varphi$ IS NORMAL, AND EVERY NORMAL SUBGROUP IS THE KER OF A HOM.)

PROP: LET $Inn(G) = \{$ INNER AUTOMORPHISMS OF $G\}$

$$= \{ \alpha_a : G \to G \mid a \in G \}$$

$$= \{ x \mapsto axa^{-1} \mid a \in G \}$$

THEN $Inn(G) \cong G/Z(G)$.

DEFINE $\psi : G/Z(G) \to Inn(G)$

$$Za \mapsto \alpha_a$$

(1) ψ IS WELL DEFINED.

SUPPOSE $Za = Zb$. MUST SHOW $\alpha_a = \alpha_b$.

IE $\alpha_a(x) = axa^{-1} = bxb^{-1} = \alpha_b(x)$.

$$Za = Zb \Rightarrow Zab^{-1} = Z \Rightarrow ab^{-1} \in Z$$

THUS $ab^{-1}x = xab^{-1} \quad \forall x \in G$.

$ba^{-1} \in Z \Rightarrow ba^{-1}x = xba^{-1}$

$$\alpha_{b^{-1}}\alpha_a(x) = \alpha_{b^{-1}}(axa^{-1}) = b^{-1}axa^{-1}b$$

$$= b^{-1}aa^{-1}bx = x.$$

THUS $\alpha_{b^{-1}} = (\alpha_a)^{-1} = \alpha_{a^{-1}}$. THUS (INVERSES ARE UNIQUE)

$\alpha_b = \alpha_a$.

- ψ IS INJECTIVE.

SUPPOSE $\psi(Za) = 1 \in \mathrm{Inn}(G)$

IE $\alpha_a(x) = axa^{-1} = x$. $\forall x \in G$.

THEN $ax = xa$. THUS $a \in Z(G)$

HENCE $Za = Z$. SO $\ker\psi = \{Z\}$.

- ψ IS SURJECTIVE. LET $\alpha_a \in \mathrm{Inn}(G)$

THEN $Za \in G/N$ AND $\psi(Za) = \alpha_a$.

QED