MATH 171: WORKSHEET 10

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ABSTRACT. An introduction to category theory following Pierre Grillet.

1. OBJECTS AND MORPHISMS

So at this point we've studied groups, rings and fields. We've seen striking similarity in some of the overarching, or underlying, structures in group theory and ring theory, for example, the isomorphism theorems. We've also seen hints at deep relationships between the algebraic objects we've studied here and objects studied elsewhere, such as geometric objects. The subject of *category theory* will help us formalize and attempt to make sense of all of this.

Definition 1. *A* category $\mathcal{C}$ *consists of*

(1) *a class* $\mathrm{Ob}(\mathcal{C})$ *whose elements are the* objects *of* $\mathcal{C}$*;*
(2) *a class* $\mathrm{Hom}(\mathcal{C})$ *whose elements are the* morphisms *or* arrows *of* $\mathcal{C}$*;*
(3) *two functions* s *and* t *which assign to each morphism of* $\mathcal{C}$ *two objects of* $\mathcal{C}$ *called the* source *or* domain *and the* target *or* codomain*; and*
(4) *a partial operation which assigns to certain pairs* (α, β) *of morphisms their* composition *or* product $\alpha\beta$ *which is a morphism in* $\mathcal{C}$

such that the following axioms hold

(a) *the composition* $\alpha\beta$ *is defined if and only if the source of* α *is the target of* β*; in that case the source of* $\alpha\beta$ *is the source of* β *and the target of* $\alpha\beta$ *is the target of* α*;*
(b) *for each object* A *in* $\mathcal{C}$ *there is an* identity morphism 1_A *whose domain and codomain are both* A *such that* $\alpha 1_A = \alpha$ *for all morphisms* α *with source* A *and* $1_A\beta = \beta$ *for any morphism* β *with* $t(\beta) = A$*;*
(c) *if* $\alpha\beta$ *and* $\beta\gamma$ *are both defined, then* $\alpha(\beta\gamma) = (\alpha\beta)\gamma$*.*

Remark 2. We use arrows to denote morphisms. The notation $\alpha : A \to B$ means that A and B are objects in cC and α is a morphism in $\mathrm{Hom}(\mathcal{C})$ and the domain or source of α is A and the target or codomain of α is B.

Example 3. (1) $\mathcal{G}\mathrm{ps}$ *The objects are groups and the morphisms are group homomorphisms.*
(2) $\mathcal{R}\mathrm{ings}$ *The objects are rings and the morphisms are ring homomorphisms.*
(3) $\mathcal{S}\mathrm{ets}$ *The objects are sets and the morphisms are maps of sets.*
(4) $\mathcal{V}\mathrm{ects}_k$ *The objects are* k*-vector spaces and the morphisms are linear transformations.*
(5) $\mathcal{A}\mathrm{b}\mathcal{G}\mathrm{ps}$ *Abelian groups.*

(6) $\mathcal{T}\mathrm{op}$ *Topological spaces and continuous maps.*
(7) *Any finite directed graph* Γ. $\mathrm{Ob}(\Gamma) = V(\Gamma)$ *and* $\mathrm{Hom}(\Gamma) = E(\Gamma)$.

Definition 4. *A category is* small *in case its classes of objects and morphisms are sets.*

Remark 5. There can be no set of all sets. Consider the contradiction brought to light by Bertrand Russell's Barber Paradox: in a small town there is a very kind barber who shaves all those (and only those) who do not shave themselves.

Definition 6. *In a category* $\mathcal{C}$, *a morphism* μ *is a* monomorphisms *if* $\mu\alpha = \mu\beta$ *implies* $\alpha = \beta$. *A morphism* σ *in* $\mathrm{Hom}(\mathcal{C})$ *is an* epimorphism *if* $\alpha\sigma = \beta\sigma$ *implies* $\alpha = \beta$. *An* isomorphism *is a morphism* $\alpha : A \to B$ *which has an inverse* $\beta : B \to A$ *such that* $\alpha\beta = 1_B$ *and* $\beta\alpha = 1_A$.

Proposition 7. *In* $\mathcal{G}\mathrm{ps}$, *a morphism is a monomorphism if and only if it is injective.*

Proof. Let $\varphi : A \to B$ be a group homomorphism. Suppose φ is injective and let α and β be homomorphisms $\alpha, \beta : C \to A$. Then $\varphi\alpha = \varphi\beta$ if and only if, for all $c \in C$,

$$\varphi(\alpha(c)) = \varphi(\beta(c)). \tag{1}$$

But φ is injective, so Equation 1 holds if and only if $\alpha(c) = \beta(c)$ for all c, i.e. $\alpha = \beta$. Hence φ is a monomorphism.

Conversely, suppose φ is a monomorphism. We show it is injective. Suppose $\varphi(x) = \varphi(y)$. Then define new morphisms

$$\alpha : \mathbb{Z} \to A \qquad\qquad \beta : \mathbb{Z} \to A$$

by

$$\alpha(1) = x \qquad\qquad \beta(1) = y.$$

Then

$$\varphi(\alpha(n)) = \varphi(\beta(n)) \text{ for all } n \in \mathbb{Z}.$$

Thus $\varphi \circ \alpha = \varphi \circ \beta$. But φ is a monomorphism, hence $\alpha = \beta$. Therefore

$$x = \alpha(1) = \beta(1) = y.$$

Therefore φ is injective. □

Proposition 8. *In* $\mathcal{G}\mathrm{ps}$, *a morphism is an epimorphism if and only if it is surjective.*

Proof. Let $\sigma : A \to B$ be a morphism in $\mathcal{C}$. Suppose σ is surjective. Then it is an epimorphism: if $\alpha, \beta : B \to C$ are morphisms such that $\alpha\sigma = \beta\sigma$, then $\alpha(b) = \beta(b)$ for all $b \in B$, since σ is surjective. Thus $\alpha = \beta$, and σ is indeed an epimorphism.

Conversely, we may show that epimorphism implies surjective. We prove the contrapositive: if σ is not surjective, then it is not an epimorphism. The elementary proof is long and the quick proof uses amalgamation, so I won't include it here. (It's in both Grillet and Riehl.)

Exercises.

(1) Show that a map of sets is a monomorphism in $\mathcal{S}\mathit{ets}$ if and only if it is injective.
(2) Show that a homomorphism of rings is a monomorphism in $\mathcal{R}\mathit{ings}$ if and only if it is injective.
(3) Show that the inclusion homomorphism $\iota : \mathbb{Z} \hookrightarrow \mathbb{Q}$ is an epimorphism in $\mathcal{R}\mathit{ings}$.