MATH 171: WORKSHEET 11

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ABSTRACT. An introduction to category theory following Pierre Grillet.

1. FUNCTORS

Can we "map" between categories? Most definitely.

Definition 1. *A* covariant functor F *from the category $\mathcal{A}$ to the category $\mathcal{B}$ assigns to each object A in $\mathcal{A}$ an object $F(A)$ in $\mathcal{B}$, and assigns to each morphism α in $\mathcal{A}$ a morphism $F(\alpha)$ in $\mathcal{B}$ so that domains, codomains, compositions, and identities are preserved; in more detail:*

(a) *if* $\alpha : A \to A'$, *then* $F(\alpha) : F(A) \to F(A')$,
(b) $F(\alpha\beta) = F(\alpha)F(\beta)$, *and*
(c) $F(1_A) = 1_{F(A)}$.

Example 2. *The* forgetful functor $F : \mathcal{R}ings \to \mathcal{G}ps$ *forgets the ring structure, i.e.*

$$F((R, +, \times)) = (R, +).$$

Remark 3. Note that there are forgetful functors between rings and sets, groups and sets, vector spaces and groups, etc.

Definition 4. *Let V be a vector space over a field k. The* Hom functor, *denoted*

$$\mathrm{Hom}(V, -) : \mathcal{V}ects_k \to \mathcal{A}b\mathcal{G}ps,$$

assigns to each vector space W the group $\mathrm{Hom}(V, W)$ of all linear transformations from V to W, and assigns to the linear tranformation $T : W \to U$ the group homomorphism

$$T_* = \mathrm{Hom}(V, T) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, U)$$

defined, for a linear tranformation $S : V \to W$ by

$$T_*(S) = T \circ S.$$

The map T_* is called the push forward *of* T.

Let's check that the Hom functor is indeed a functor. First, we need to show that, for any k-vector space W, the set $\mathrm{Hom}(V, W)$ is indeed an abelian group. Let $S_1, S_2 : V \to W$ be linear transformation. Then define $S_1 + S_2$ by

$$(S_1 + S_2)(v) = S_1(v) + S_2(v) \text{ for all } v \in V.$$

Then $S_1 + S_2$ is indeed a linear transformation, as

$$\begin{aligned}(S_1 + S_2)(x + y) &= S_1(x + y) + S_2(x + y) \\ &= S_1(x) + S_1(y) + S_2(x) + S_2(y) \\ &= S_1(x) + S_2(x) + S_1(y) + S_2(y) \\ &= S_1 + S_2(x + y).\end{aligned}$$

Similarly for scalar multiplication. Thus $\mathrm{Hom}(V, W)$ is an abelian group.

Now, we must check that, for any linear transformation $T : W \to U$, $T_* = \mathrm{Hom}(V, T) : \mathrm{Hom}(V, W) \to \mathrm{Hom}(V, U)$ is a homomorphism of abelian groups. So, let $S_1, S_2 : V \to W$ be linear transformations. Then

$$\begin{aligned}T_*(S_1 + S_2)(v) &= T \circ (S_1 + S_2)(v) \\ &= T((S_1(v) + S_2(v))) \\ &= T(S_1(v)) + T(S_2(v)) \text{ as } T \text{ is a linear transformation} \\ &= T_*(S_1)(v) + T_*(S_2)(v).\end{aligned}$$

Thus

$$T_*(S_1 + S_2) = T_*S_1 + T_*S_2.$$

Therefore T_* is a group homomorphism which respects domains and codomains (Part (a) of our definition).

Also, if $T : W \to U$ and $L : U \to Z$ are linear transformations, then, for $S : V \to W$

$$\begin{aligned}\mathrm{Hom}(V, L \circ T)(S) &= (L \circ T)_*(S) \\ &= (L \circ T) \circ S \\ &= L \circ T \circ S \\ &= L_*T_*(S) \\ &= \mathrm{Hom}(V, L) \circ \mathrm{Hom}(V, T)(S).\end{aligned}$$

Thus $\mathrm{Hom}(V, L \circ T) = \mathrm{Hom}(V, L) \circ \mathrm{Hom}(V, T)$. Thus (b) is satisfied (the Hom functor is indeed functorial).

Lastly, for any $S \in \mathrm{Hom}(V, W)$, we compute

$$\begin{aligned}\mathrm{Hom}(V, 1_W)(S) &= (1_W)_*(S) \\ &= 1_V \circ S \\ &= S \\ &= 1_{\mathrm{Hom}(V,W)}(S).\end{aligned}$$

Thus $\mathrm{Hom}(V, 1_W) = 1_{\mathrm{Hom}(V,W)}$, and (c) holds. Therefore $\mathrm{Hom}(V, -)$ is indeed a functor.

2. Natural Tranformations

Can we map between functors? Of course!

Definition 5. *Let* F *and* G *be functors from* $\mathcal{A}$ *to* $\mathcal{B}$. *A natural transformation* $\tau : \mathsf{F} \to \mathsf{G}$ *assigns to each object* A *in* $\mathcal{A}$ *a morphism* $\tau_{\mathsf{A}} : \mathsf{F}(\mathsf{A}) \to \mathsf{G}(\mathsf{A})$ *in* $\mathcal{B}$ *such that, for each morphsim* $\alpha : \mathsf{A}_1 \to \mathsf{A}_2$ *in* $\mathcal{A}$*, the following diagram commutes:*

$$\begin{array}{ccc} \mathsf{F}(\mathsf{A}_1) & \xrightarrow{\tau_{\mathsf{A}_1}} & \mathsf{G}(\mathsf{A}_1) \\ {\scriptstyle \mathsf{F}(\alpha)}\downarrow & & \downarrow{\scriptstyle \mathsf{G}(\alpha)} \\ \mathsf{F}(\mathsf{A}_2) & \xrightarrow[\tau_{\mathsf{A}_2}]{} & \mathsf{G}(\mathsf{A}_2) \end{array}$$