MATH 171: WORKSHEET R2

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ABSTRACT. The subjects of Class R2 are ring homomorphisms, subrings and quotients rings. This corresponds roughly with Section 4.3 of Herstien.

Definition 1. *A* subring S *of a ring* $(R, +, \cdot)$ *is a subgroup* $S \leqslant (R, +)$ *which is closed under the multiplicative structure of* R*, i.e.*

$$a \cdot b \in S \quad \textit{for all } a, b \in S.$$

Example 2. $\mathbb{Q}$ *is a subring of* $\mathbb{R}$.

Proposition 3. *A subset* S *of the ring* R *is a subring if and only if* S *is closed under subtraction and multiplication.*

Proof. This follows immediately from the fact that a subset H of an Abelian group G is a subgroup if and only if H is closed under subtraction. $\square$

Example 4. *The* center *of a ring* A *is the set of elements* $a \in A$ *such that* $ax = xa$ *for all* $x \in A$. *The center of* A *is a subring of* A.

Definition 5. *Let* R *and* S *be rings. A* ring homomorphism *is a map of sets* $\varphi : R \to S$ *such that, for all* $a, b \in R$,

(1)
$$\varphi(a + b) = \varphi(a) + \varphi(b)$$

(2)
$$\varphi(ab) = \varphi(a) \cdot \varphi(b)$$

Definition 6. *The* kernel *of the homomorphism* $\varphi : R \to S$ *is given by*

$$\operatorname{Ker}(\varphi) = \{r \in R : \varphi(r) = 0 \in S\}.$$

Definition 7. *An* isomorphism *of rings is a bijective homomorphism.*

Definition 8. *A subring* I *of* R *is a* left ideal *of* R *if* I *is closed under left multiplication by elements from* R*, i.e.* $rI \subset I$ *for all* $r \in R$. *Similarly,* I *is a* right ideal *of* R *if* I *is closed under right multiplication by elements of* R*, i.e.* $Ir \subset R$ *for all* $r \in R$. *A subring which is both a left and right ideal is called a* two sided ideal*, or simply* ideal.

Definition 9. *The* quotient ring R/I *of the ring* R *by the ideal* $I \subset R$ *is the quotient group of cosets* R/I *under the operations*

$$(r + I) + (s + I) = (r + s) + I \qquad\qquad (r + I) \cdot (s + I) = (r \cdot s) + I,$$

for all $r, s \in I$.

Proposition 10. *For any ring* R *and ideal* I, R/I *is a ring.*

Theorem 11 (First isomporphism theorem for rings)**.** *If* $\varphi : R \to S$ *is a homomorphism of rings, then* $\operatorname{Ker}(\varphi)$ *is an ideal of* R, $\operatorname{Im}(\varphi)$ *is a subring of* S, *and*

$$R/\operatorname{Ker}(\varphi) \cong \operatorname{Im}(\varphi).$$

Furthermore, if I *is any ideal of* R, *the map*

$$R \to R/I$$

defined by $r \mapsto r + I$ *is a surjective ring homomorphisms with kernel* I.

Theorem 12 (Second isomorphism theorem for rings)**.** *Let* R *be a ring,* A *a subring and* B *an ideal of* R. *Then*

$$A + B = \{a + b : a \in A,\ b \in B\}$$

is a subring of R, $A \cap B$ *is an ideal of* A *and*

$$(A + B)/B \cong A/(A \cap B).$$

Theorem 13 (Third isomorphism theorem for rings)**.** *Let* I *and* J *be ideals of the ring* R *such that* $I \subset J$. *Then* J/I *is an ideal of* R/I *and*

$$(R/I)/(J/I) \cong R/J.$$

Theorem 14 (Fourth isomorphism theorem for rings)**.** *Let* I *be an ideal of* R. *The correspondence*

$$A \longleftrightarrow A/I$$

is an inclusion preserving bijection between the subrings A *of* R *containing* I *and the set of subrings of* R/I. *Further, a subring* A *containing* I *is an ideal of* R *if and only if* A/I *is an ideal of* R/I.

0.1. **Exercise.** Complete the following exercises as a group.

(1) Let $\mathbb{Z}[\sqrt{2}] = \{a + b\sqrt{2} : a, b \in \mathbb{Z}\}$, and let

$$H = \left\{ \begin{pmatrix} a & 2b \\ b & a \end{pmatrix} : a, b \in \mathbb{Z} \right\}.$$

- (a) Show that $\mathbb{Z}[\sqrt{2}]$ is a ring.
- (b) Show that H is a ring.
- (c) Show that H and $\mathbb{Z}[\sqrt{2}]$ are isomorphic as rings.

(2) Let $R = \mathbb{Z}[x]$, and I be all multiples of x^2, i.e.

$$I = \{f \in \mathbb{Z}[x] \mid f = x^2 \cdot g(x) \text{ for some polynomial } g(x) \in \mathbb{Z}[x]\}.$$

- (a) Show that I is a subring of R.
- (b) Show that I in an ideal of R.
- (c) Describe R/I.

(3) (a) Show that $2\mathbb{Z}$ and $3\mathbb{Z}$ are not isomorphic (as rings).

(b) Show that $\mathbb{Z}[x]$ and $\mathbb{Q}[x]$ are not isomorphic (as rings).

(c) Is the map from $\mathbb{Z}/10\mathbb{Z}$ to $\mathbb{Z}/10\mathbb{Z}$ given by $x \mapsto 2x$ a ring homomorphism?

(4) (a) Determine all the ring homomorphisms from $\mathbb{Z}$ to $\mathbb{Z}$.

(b) Determine all ring homomorphisms from $\mathbb{Q}$ to $\mathbb{Q}$.

(c) Determine all ring homomorphisms from $\mathbb{Z}_{20}$ to $\mathbb{Z}_{30}$.

(5) Let A and B be ideals of a ring R. The *product* of A and B is given by

$$AB = \{a_1 b_1 + \cdots + a_n b_n : a_i \in A,\ b_i \in B,\ n \in \mathbb{N}\}.$$

(a) Show that AB is an ideal.

(b) Show $AB \subset A \cap B$

(c) If R is commutative with 1 and $A + B = R$, then show $A \cap B = AB$.

(6) (a) $\mathrm{Ker}(\varphi)$ is an ideal and $\mathrm{Im}\,\varphi$ is a subring.

(b) $R/\operatorname{Ker}\varphi \cong \mathrm{Im}\,\varphi$

(c) The map $R \to R/I$ is a surjective ring homomorphism.

(7) Let R be a commutative ring with 1, and $G = \{g_1, \ldots, g_n\}$ be a finite group. The *group ring* RG of G over R is the set of all formal linear combinations of elements of G with coefficients in R, i.e.

$$RG = \{r_1 g_1 + r_2 g_2 + \cdots + r_n g_n \mid n \in \mathbb{N},\ r_i \in R,\ g_i \in G\}.$$

For example,

$$5(12) + 3(12) + 2 \cdot (7(123)) = 8(12) + 14(123) \in \mathbb{Z}S_3.$$

The map $RG \to R$ given by

$$\sum_{i=1}^{n} a_i g_i \mapsto \sum_{i=1}^{n} a_i$$

is called the *augmentation map*.

(a) Show that the augmentation map is a ring homomorphism

(b) Describe the kernel of the augmentation map.

(c) Show that the associated quotient ring is isomorphic to R.