MATH 171: WORKSHEET R4

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1. Properties of ideals

Throughout these notes, let R be a ring with 1.

Definition 1. *Let $A \subseteq R$ be a subset. Then* the (left) ideal generated by A *is the smallest (left) ideal of R containing A, and is denoted (A).*

Note 2. *Recall the meaning of "smallest" in such context; (A) is the smallest ideal in R containing A if and only if and ideal which contains A also contains (A), i.e. for an ideal J of R,*

$$A \subseteq J \Rightarrow (A) \subseteq J.$$

Remark 3. Note that (A) is the intersection of all ideals I containing A.

$$(A) = \bigcap_{A \subseteq I} I$$

Indeed, R is an ideal of itself containing A, and the intersection of nonempty ideals is an ideal. By definition the intersection contains A. Therefore $\bigcap_{A\subseteq I} I$ is an ideal containing A. Since (A) is the smallest ideal containing A, we conclude $(A) \subseteq \bigcap_{A\subseteq I} I$.

On the other hand, suppose $x \in \bigcap_{A\subseteq I} I$. Then for any ideal I containing A, $x \in I$. But (A) is an ideal containing A. Thus $x \in (A)$. Therefore $\bigcap_{A\subseteq I} I \subseteq (A)$. Thus we have shown $(A) = \bigcap_{A\subseteq I} I$.

Definition 4. *We define the notation RA by:*

$$RA = \{r_1 a_1 + r_2 a_2 + \cdots + r_n a_n \mid r_i \in R,\ a_i \in A,\ n \in \mathbb{Z}\}.$$

Similarly for AR.

Proposition 5. *If R is commutative, then $(A) = RA$.*

Remark 6. If R is commutative, then

$$AR = RA = (A).$$

Definition 7. *An ideal generated by a single element is called an* principal ideal*, and an ideal generated by a finite set is called a* finitely generated ideal*. An ideal I of a ring R is* proper *if it is a proper subset, i.e. $I \neq R$ i.e. $I \subsetneq R$.*

Definition 8. *A proper ideal M of a ring R is* maximal *if whenever I is an ideal of R and $M \subset I \subset R$, then $M = I$ or $I = R$.*

Definition 9. *A proper ideal P of a commutative ring R is* prime *if $ab \in P$ implies $a \in P$ or $b \in P$ for any $a, b \in R$.*

2. EXERCISES

2.1. **Exercise.** Prove Proposition 5 in the following steps.

(1) Show that RA is closed under addition and left multiplication by element of R.
(2) Show that RA contains A.
(3) Conclude that RA is an ideal containing A.
(4) Suppose that J is an ideal containing A. Show that $RA \subset J$.
(5) Conclude that RA is the left ideal generated by A.

2.2. **Exercise.** Consider the finitely generated ideal $(2, x)$ in $\mathbb{Z}[x]$.

(1) What do the elements of $(2, x)$ look like?
(2) Note that $(2, x)$ is a proper ideal, i.e. $(2, x) \neq \mathbb{Z}[x]$.
(3) Show that $(2, x)$ is not a principal ideal. Hint: Suppose otherwise, then $(2, x) = (a(x))$ for some $a(x) \in \mathbb{Z}[x]$.

2.3. **Exercise.**

(1) Show that the ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is prime if and only if n is prime.
(2) Inspect the lattice of subgroups of $\mathbb{Z}/36\mathbb{Z}$ and show that (2) and (3) are maximal ideals.
(3) Show that the ideal $(x^2 + 1)$ is not prime in $\mathbb{Z}_2[x]$.