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ALGEBRAIC GEOMETRY CLASS 7

RECALL: LET R BE A COMMUTATIVE RING W/ 1. AN IDEAL M OF R IS MAXIMAL IF, FOR AN IDEAL J OF R,

$$M \subseteq J \subseteq R \Rightarrow M = J \text{ OR } J = R,$$

IE THERE ARE NO PROPER IDEALS OF R STRICTLY CONTAINING M AS A PROPER SUBSET. AN IDEAL P OF R IS PRIME IF $fg \in P \Rightarrow f \in P$ OR $g \in P$. AN IDEAL $I \subseteq R$

IS MAXIMAL $\Leftrightarrow$ R/I IS A FIELD AND

IS PRIME $\Leftrightarrow$ R/I IS AN INTEGRAL DOMAIN.

FIELDS ARE INTEGRAL DOMAINS, SO MAX'L IDEALS ARE PRIME.

(DO YOU RECALL HOW TO PROVE THIS DIRECTLY?)

THE RADICAL OF I IS $\text{RAD}(I) = \sqrt{I} = \{f \in R \mid f^n \in I \text{ FOR SOME } n \in \mathbb{N}\}$
AN IDEAL IS RADICAL IF $I = \sqrt{I}$.

NOTE: PRIME $\Rightarrow$ RADICAL

PF: WE ASSUME I IS PRIME AND SHOW

$$I \subseteq \sqrt{I} \text{ AND } \sqrt{I} \subseteq I.$$

LET $f \in I$. THEN $f \in \sqrt{I}$ $(n=1)$. HENCE $I \subseteq \sqrt{I}$

NOW SUPPOSE $g \in \sqrt{I}$. THEN $\exists\ n \in \mathbb{N}$ SUCH THAT

$$g^n \in I.$$

BUT I IS PRIME. HENCE $g \in I$ OR $g^{n-1} \in I$. AS I IS PRIME.

IF $g \in I$ WE ARE DONE. OTHERWISE $g^{n-2} \in I$. REPEATING

YIELDS $g \in I$. THUS $\sqrt{I} \subseteq I$. QED.

NULLSTELLENSATZ (HILBERT: ZEROS THM)

LET k BE ALGEBRAICALLY CLOSED, $k = \bar{k}$.

(EVERY NONCONSTANT POLYNOMIAL IN $k[x]$ HAS A ROOT IN k)

(a) EVERY MAXIMAL IDEAL OF $A = k[x_1, .., x_n]$ IS OF THE FORM $m_p = (x_1 - a_1, x_2 - a_2, .., x_n - a_n)$ FOR SOME POINT $p = (a_1, .., a_n) \in \mathbb{A}^n_k$, IE $m_p = I(p)$.

(b) LET $J \subseteq A$ BE AN IDEAL. $J \neq (1)$ IMPLIES $Z(J) \neq \emptyset$.

(c) $$I(Z(J)) = \sqrt{J}.$$

REMARK! $(1) = A$. SO (b) IS SAYING ANY COLLECTION OF POLYNOMIALS THAT DON'T GENERATE ALL OF A HAVE A COMMON ZERO.

COROLLARY.

NOT BIJECTIONS

$$\begin{array}{ccc} \{\text{IDEALS OF } A\} & \underset{Z}{\overset{I}{\rightleftarrows}} & \{\text{SUBSETS OF } \mathbb{A}^n_k\} \\ \cup & & \cup \\ \{\text{RADICAL IDEALS}\} & \longleftrightarrow & \{\text{ALGEBRAIC SUBSETS}\} \\ \cup & \text{BIJECTIONS} & \cup \\ \{\text{PRIME IDEALS}\} & \longleftrightarrow & \{\text{AFFINE ALGEBRAIC VARIETIES}\} \end{array}$$

GROUPWORK!

(REID 3.5) LET $J = (xy, xz, yz) \subseteq k[x,y,z]$. FIND $Z(J)$.

IS $Z(J)$ IRREDUCIBLE? $J = I(Z(J))$?

LET $J' = (xy, (x-y)z)$. FIND $Z(J')$, $\sqrt{J'}$.

(REID 3.6) LET $J = (x^2 + y^2 - 1,\ y - 1)$. FIND $f \in I(Z(J)) \setminus J$.

(REID 3.7) LET $J = (x^2 + y^2 + z^2,\ xy + xz + yz)$.

FIND $Z(J)$ AND $I(Z(J))$.