In this lecture, we provide one interpretation of a matrix, giving the notion of matrix greater depth than just an array of numbers.

Definition 1. A linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) is a map \( T : \mathbb{R}^n \to \mathbb{R}^m \) such that

1. \( T(c\vec{v}) = cT(\vec{v}); \) and
2. \( T(\vec{u} + \vec{v}) = T(\vec{u}) + T(\vec{v}), \) for any vectors \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and scalar \( c. \)

Remark 2. \( T \) is a linear transformation if and only if

\[
T(a\vec{u} + b\vec{v}) = aT(\vec{u}) + bT(\vec{v}),
\]

for any vectors \( \vec{u}, \vec{v} \in \mathbb{R}^n \) and any scalars \( a \) and \( b. \)

Example 3. Let \( T : \mathbb{R}^n \to \mathbb{R}^n \) be given by

\[
T(\vec{v}) = 5\vec{v}.
\]

Then for any scalar \( c, \)

\[
T(c\vec{v}) = 5c\vec{v} = c5\vec{v} = cT(\vec{v})
\]

and

\[
T(\vec{u} + \vec{v}) = 5(\vec{u} + \vec{v}) = 5\vec{u} + 5\vec{v} = 5T(\vec{u}) + 5T(\vec{v}).
\]

Therefore \( T \) is a linear transformation.

Example 4. Let \( F : \mathbb{R}^2 \to \mathbb{R}^2 \) be given by

\[
F(x_1, x_2) = (\sin(x_1), e^{x_2}).
\]

On the one hand,

\[
F(0, 0) + F(\pi/2, 0) = (0, 1) + (1, 1)
= (0, 2).
\]

On the other hand

\[
F((0, 0) + (\pi/2, 0)) = F(\pi/2, 0)
= (0, 1).
\]

Since \( (0, 1) \neq (0, 2), \) \( F \) is not linear.

Definition 5. Let \( A \) be an \( m \times n \) matrix. The matrix transformation \( T_A \) of \( A \) is the map \( T_A : \mathbb{R}^n \to \mathbb{R}^m \) defined by

\[
T_A(\vec{x}) = A\vec{x},
\]

for any \( \vec{x} \in \mathbb{R}^n. \)

Proposition 6. For any matrix \( A, \) \( T_A \) is a linear transformation.
Let \( c \) be a scalar, and \( \vec{u}, \vec{v} \in \mathbb{R}^n \). Then
\[
T_A(c \vec{v}) = A(c \vec{v}) = cA\vec{v} = cT_A(\vec{v}),
\]
and
\[
T_A(\vec{u} + \vec{v}) = A(\vec{u} + \vec{v}) = A\vec{u} + A\vec{v} = T(\vec{u}) + T(\vec{v}).
\]
\[\square\]

**Theorem 7.** Let \( T : \mathbb{R}^n \rightarrow \mathbb{R}^m \) be a linear transformation. Then \( T \) is the matrix transformation of the \( m \times n \) matrix \( A \) given by
\[
A = (T(\vec{e}_1) \ T(\vec{e}_2) \ \cdots \ T(\vec{e}_n)).
\]

**Proof.** Let \( \vec{x} \in \mathbb{R}^n \). Then
\[
\vec{x} = (x_1, \ldots, x_n) = x_1\vec{e}_1 + \cdots + x_n\vec{e}_n.
\]
Therefore
\[
T(\vec{x}) = T(x_1\vec{e}_1 + \cdots + x_n\vec{e}_n) = T(x_1\vec{e}_1) + \cdots + T(x_n\vec{e}_n) = x_1T(\vec{e}_1) + \cdots + x_nT(\vec{e}_n)
\]
\[
= (T(\vec{e}_1) \ T(\vec{e}_2) \ \cdots \ T(\vec{e}_n)) \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{pmatrix}
\]
\[
= A\vec{x}.
\]

**Definition 8.** \( T_A \) is called the standard matrix of \( T \) and is denoted \([T]\).

**Example 9.** If \( T(\vec{v}) = 5\vec{v} \), then \([T] = 5I_n\), since \( T(\vec{e}_i) = 5\vec{e}_i \).

**Theorem 10.** If \( S \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^m \) and \( T \) is a linear transformation from \( \mathbb{R}^m \) to \( \mathbb{R}^k \), then \( S \circ T \) is a linear transformation from \( \mathbb{R}^n \) to \( \mathbb{R}^k \) and
\[
[S \circ T] = [S][T].
\]

**Definition 11.** Let \( S, T : \mathbb{R}^n \rightarrow \mathbb{R}^n \) be linear transformations. Then \( S \) and \( T \) are inverse transformations if
\[
S \circ T = I_n = T \circ S.
\]
We say that \( S \) and \( T \) are invertible transformations.

**Theorem 12.** Let \( T \) be an invertible linear transformation. Then \([T]\) is an invertible matrix, and
\[
[T^{-1}] = [T]^{-1}.
\]

**Definition 13.** An invertible linear transformation is called an isomorphism.

**Theorem 14.** For any linear transformation,
\[
T(\vec{0}) = \vec{0}.
\]

**Proof.**
\[
T(\vec{0}) = T(\vec{0} + \vec{0}) = T(\vec{0}) + T(\vec{0}).
\]
\[\square\]
Remark 15. Why are linear transformations called linear transformations? Perhaps this is easiest to see in $\mathbb{R}^1$. Let $T : \mathbb{R} \to \mathbb{R}$ be a linear transformation. The it is of the form $T(x) = ax$ for some scalar $a$. This is automatic from the theorem above, since any $1 \times 1$ matrix is simply a scalar.

But we can also derive this result from first principles. Let $T(1) = a$.

Then

$$a = T(1) = T \left( \left( \frac{1}{x} \right) x \right) = \frac{1}{x} T(x).$$

Thus $T(x) = a(x)$ for all $x \neq 0$. But $T(0) = 0$ by the above, and thus $T(0) = 0 = 5 \cdot 0$. Thus $T(x) = ax$ for all $x \in \mathbb{R}$. Thus the graph of $T$ is a line through the origin of slope $a$. 

3