Spanning sets and linear independence

**Span (continued)**

**Summary**

Given $\mathbf{v}_1, \ldots, \mathbf{v}_k, \mathbf{b} \in \mathbb{R}^n$,

$\mathbf{b} \in \text{span}(\mathbf{v}_1, \ldots, \mathbf{v}_k) \iff \mathbf{b}$ is a linear comb. of $\mathbf{v}_1, \ldots, \mathbf{v}_k$ \iff $[A | \mathbf{b}]$ has a solution, where $A$ is matrix w/columns as $\mathbf{v}_1, \ldots, \mathbf{v}_k$

Given $\mathbf{v}_1, \ldots, \mathbf{v}_k \in \mathbb{R}^n$,

$S = \{\mathbf{v}_1, \ldots, \mathbf{v}_k\}$ is a spanning set of $\mathbb{R}^n \iff \text{span}(S) = \mathbb{R}^n \iff [A | \mathbf{b}]$ has a solution for any $\mathbf{b} \in \mathbb{R}^n$

**Linear independence**

**Example**  Recall we previously found that

$$\begin{pmatrix} -3 \\ 8 \\ -5 \end{pmatrix} = 3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -1 \\ 1 \end{pmatrix}.$$

Rearranging terms, we have

$$3 \begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix} - 2 \begin{pmatrix} 3 \\ -1 \\ 8 \end{pmatrix} - \begin{pmatrix} -3 \\ -5 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}.$$

Thus, we have a nontrivial way to express $\mathbf{0}$ as a linear combination of $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$. This is the definition of linear dependence.
**Definition** We say vectors \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) are *linearly dependent* if \( \exists \) scalars \( c_1, c_2, \ldots, c_k \), not all zero, such that

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0}. \quad (\ast)
\]

Otherwise, the vectors are *linearly independent*, which means the only solution to (\ast) is the trivial solution \( c_1 = \cdots = c_k = 0 \).

To determine if \( \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \) are linearly independent or not, we need to know if \( \exists \) nontrivial solution to

\[
c_1 \vec{v}_1 + c_2 \vec{v}_2 + \cdots + c_k \vec{v}_k = \vec{0} \quad \implies \quad \begin{bmatrix} c_1 & c_2 & \cdots & c_k \end{bmatrix} \begin{bmatrix} \vec{v}_1 & \vec{v}_2 & \cdots & \vec{v}_k \end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix}
\]

This is a homogeneous linear system! We need to determine if the system has one solution or more than one solution.

\( \vec{v}_1, \ldots, \vec{v}_k \) are linearly independent \( \iff \) \([ A \mid \vec{0}] \) has a unique solution, namely \( \vec{0} \)

(where \( A \) has \( \vec{v}_1, \ldots, \vec{v}_k \) as columns)

**Example** Are the following vectors linearly independent?

\[
\begin{bmatrix} 0 \\ 1 \\ 5 \end{bmatrix}, \begin{bmatrix} 1 \\ 2 \\ 8 \end{bmatrix}, \begin{bmatrix} 4 \\ -1 \\ 0 \end{bmatrix}
\]

We do EROs on the appropriate augmented matrix to get

\[
\begin{bmatrix} 0 & 1 & 4 & 0 \\ 1 & 2 & -1 & 0 \\ 5 & 8 & 0 & 0 \end{bmatrix} \xrightarrow{\text{EROs}} \begin{bmatrix} 1 & 2 & -1 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 13 & 0 \end{bmatrix}.
\]
We see that there are no free variables (i.e., every column has a leading entry), so the system has a unique solution, \( \vec{0} \). Thus, the vectors are linearly independent.

**Theorem.**

Vectors \( \vec{v}_1, \ldots, \vec{v}_k \) are linearly dependent if and only if one of the vectors can be written as a linear combination of the others.

**Facts**  Let \( S = \{ \vec{v}_1, \vec{v}_2, \ldots, \vec{v}_k \} \) be a set of vectors in \( \mathbb{R}^n \).

1. If \( \vec{v}_i = \vec{0} \) for some \( i \), then \( S \) is linearly dependent.
2. If one vector in \( S \) is a linear combination of the other vectors in \( S \), then \( S \) is linearly dependent.
3. If \( k > n \) (more vectors than components), then \( S \) is linearly dependent.

Homogeneous system w/more variables than equations must have a free variable.

**Example**  Is \( \left\{ \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \begin{bmatrix} -2 \\ 0 \\ 1 \end{bmatrix}, \begin{bmatrix} -10 \\ 21 \\ -5 \end{bmatrix}, \begin{bmatrix} 0 \\ -\frac{1}{7} \end{bmatrix} \right\} \) linearly independent?

**No!** The vectors are linearly dependent since we have 4 vectors in \( \mathbb{R}^3 \). (See fact 3 above.)
Matrices

At its core, linear algebra is the study of linear transformations and their algebraic properties. We’ll see, down the road, that there is an intimate relationship between a linear transformation and a matrix.

Recall...

**Definition** A *matrix* is a rectangular array of numbers.

**Example** $A = \begin{bmatrix} 1 & 2 \\ \frac{-3}{2} & 3 \\ 0 & -5 \end{bmatrix}$ is a $3 \times 2$ matrix.

$a_{ij}$ denotes the entry of $A$ in row $i$ and column $j$, so, for example, $a_{12} = 2$ and $a_{21} = \frac{-3}{2}$.

**Definition** If $A$ is an $n \times n$ matrix (i.e., # of rows = # of cols.), then we say that $A$ is a *square matrix*.

**Matrix operations**

- **Equality:**
  \[ A = B \iff A, B \text{ are same size and } a_{ij} = b_{ij} \forall i, j \]

- **Matrix addition:** $A, B$ are $m \times n$ matrices
  \[ C = A + B \text{ is the } m \times n \text{ matrix defined as } c_{ij} = a_{ij} + b_{ij} \forall i, j \]

  add entrywise

- **Scalar multiplication:** $m \times n$ matrix $A$, scalar $c$
  \[ cA \text{ is the } m \times n \text{ matrix with entries } ca_{ij} \forall i, j \]
Example

\[ 2\begin{bmatrix} 2 & \frac{5}{2} & 1 \\ -1 & 0 & 2 \end{bmatrix} + \begin{bmatrix} 3 & 1 & -4 \\ 1 & 5 & 2 \end{bmatrix} = \begin{bmatrix} 7 & 6 & -2 \\ -1 & 5 & 6 \end{bmatrix} \]

Remark  The set of all \( m \times n \) matrices with real entries (denoted \( \mathbb{R}^{m \times n} \) or \( M_{m \times n}(\mathbb{R}) \)) with the operations of matrix addition and scalar multiplication form a vector space.

\[ A, B, C \text{ are } m \times n \text{ matrices, } c, d \text{ are scalars} \]

(1) \( A + B \) is an \( m \times n \) matrix \hspace{1cm} \text{(closure under addition)}
(2) \( A + B = B + A \) \hspace{1cm} \text{(commutativity)}
(3) \( (A + B) + C = A + (B + C) \) \hspace{1cm} \text{(associativity)}
(4) \( A + 0 = A \) \hspace{1cm} \text{(existence of additive identity)}
(5) \( A + (-A) = 0 \) \hspace{1cm} \text{(existence of additive inverses)}
(6) \( cA \) is an \( m \times n \) matrix \hspace{1cm} \text{(closure under scalar multiplication)}
(7) \( c(A + B) = cA + cB \) \hspace{1cm} \text{(distributivity)}
(8) \( (c + d)A = cA + dA \) \hspace{1cm} \text{(distributivity)}
(9) \( c(dA) = (cd)A \)
(10) \( 1A = A \)

Matrix multiplication \( \Longrightarrow \) see slides
Matrix multiplication

Math 40, Introduction to Linear Algebra
Monday, January 30, 2012

Matrix-vector multiplication: two views

- 1st perspective: \( Ax \bar{x} \) is linear combination of columns of \( A \)

\[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
4 \\
3 \\
2 \\
\end{bmatrix}
= 4 \begin{bmatrix}
1 \\
2 \\
\end{bmatrix}
\]
Matrix-vector multiplication: two views

- 1st perspective: \( A \vec{x} \) is linear combination of columns of \( A \)

\[
\begin{bmatrix}
  1 & -2 & 3 \\
  2 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
  4 \\
  3 \\
  2 \\
\end{bmatrix}
= 4 \begin{bmatrix}
  1 \\
  2 \\
\end{bmatrix}
+ 3 \begin{bmatrix}
  -2 \\
  1 \\
\end{bmatrix}
+ 2 \begin{bmatrix}
  3 \\
  5 \\
\end{bmatrix}
= \begin{bmatrix}
  4 \\
  21 \\
\end{bmatrix}
\]

- 2nd perspective: \( A \vec{x} \) is computed as dot product of rows of \( A \) with vector \( \vec{x} \)

\[
\begin{bmatrix}
  1 & -2 & 3 \\
  2 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
  4 \\
  3 \\
  2 \\
\end{bmatrix}
= \text{dot product of } \begin{bmatrix}
  1 \\
  -2 \\
\end{bmatrix}
\text{ and } \begin{bmatrix}
  4 \\
  3 \\
\end{bmatrix}
= \begin{bmatrix}
  4 \\
  21 \\
\end{bmatrix}
\]

Notice that \# of columns of \( A \) = \# of rows of \( \vec{x} \).
This is a requirement in order for matrix multiplication to be defined.
Matrix multiplication (in general)

\[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 1 & 5
\end{bmatrix}
\begin{bmatrix}
4 & 2 & 1 \\
3 & 0 & 2 \\
2 & 1 & 3
\end{bmatrix}
\]

\[A \quad B\]

Note that
\[\# \text{ cols. of } A = \# \text{ of rows of } B\]

\[AB = \begin{bmatrix}
A \\
\vec{b}_1 \\
\vec{b}_2 \\
\vdots \\
\vec{b}_p
\end{bmatrix}
= \begin{bmatrix}
A\vec{b}_1 \\
A\vec{b}_2 \\
\vdots \\
A\vec{b}_p
\end{bmatrix}\]

\[\begin{array}{c}
m \times n \\
n \times p \\
m \times p
\end{array}\]

Each column of \(AB\) is a linear combination of columns of \(A\).

Matrix multiplication (in general)

\[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 1 & 5
\end{bmatrix}
\begin{bmatrix}
4 & 2 & 1 \\
3 & 0 & 2 \\
2 & 1 & 3
\end{bmatrix}
= \begin{bmatrix}
4 \quad 5 \quad 6 \\
21 \quad 9 \quad 19
\end{bmatrix}
\]

Computing \(AB\) via linear combinations of columns of \(A\):

1st column of \(AB\) = \(4\left[\begin{array}{c}1 \\ 2\end{array}\right] + 3\left[\begin{array}{c}2 \\ 1\end{array}\right] + 2\left[\begin{array}{c}1 \\ 3\end{array}\right] = \left[\begin{array}{c}4 \\ 21\end{array}\right]\)

2nd column of \(AB\) = \(2\left[\begin{array}{c}1 \\ 2\end{array}\right] + 0\left[\begin{array}{c}2 \\ 1\end{array}\right] + 1\left[\begin{array}{c}3 \\ 5\end{array}\right] = \left[\begin{array}{c}5 \\ 9\end{array}\right]\)

3rd column of \(AB\) = \(1\left[\begin{array}{c}1 \\ 2\end{array}\right] + 2\left[\begin{array}{c}2 \\ 1\end{array}\right] + 3\left[\begin{array}{c}3 \\ 5\end{array}\right] = \left[\begin{array}{c}6 \\ 19\end{array}\right]\)

While you should understand this approach, it is often easier to multiply matrices via dot products.
Matrix multiplication (in general)

In terms of dot products,
the \((i,j)\)-entry of \(AB\) = \([ith \ row \ of \ A]\cdot [jth \ column \ of \ B]\)

viewed as column vectors

\[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
4 & 2 & 1 \\
3 & 0 & 2 \\
2 & 1 & 3 \\
\end{bmatrix}
= \begin{bmatrix}
4 \\
\end{bmatrix}
\]

since \[
\begin{bmatrix}
1 \\
-2 \\
3 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
4 \\
3 \\
2 \\
\end{bmatrix}
= 4
\]

Matrix multiplication (in general)

In terms of dot products,
the \((i,j)\)-entry of \(AB\) = \([ith \ row \ of \ A]\cdot [jth \ column \ of \ B]\)

viewed as column vectors

\[
\begin{bmatrix}
1 & -2 & 3 \\
2 & 1 & 5 \\
\end{bmatrix}
\begin{bmatrix}
4 & 2 & 1 \\
3 & 0 & 2 \\
2 & 1 & 3 \\
\end{bmatrix}
= \begin{bmatrix}
4 & 5 & 6 \\
21 & 9 & 19 \\
\end{bmatrix}
\]

since \[
\begin{bmatrix}
2 \\
1 \\
5 \\
\end{bmatrix}
\cdot
\begin{bmatrix}
1 \\
2 \\
3 \\
\end{bmatrix}
= 19
\]
Matrix multiplication

What sizes of matrices can be multiplied together?

For $m \times n$ matrix $A$ and $n \times p$ matrix $B$, the matrix product $AB$ is an $m \times p$ matrix.

If $A$ is a square matrix and $k$ is a positive integer, we define

$$ A^k = A \cdot A \cdots A $$

$k$ factors