Similarity and diagonalizability

Assume throughout today’s discussion that all matrices are square \((n \times n)\).

**Definition** A matrix \(A\) is *similar* to \(B\) if there exists an invertible matrix \(P\) such that \(P^{-1}AP = B\). (We denote as \(A \sim B\).)

**Remarks** Similar matrices have the same
- determinant,
  \[
  \det B = \det(P^{-1}AP) = \det(P^{-1}) \det A \det P = \det A \frac{1}{\det P} \det P = \det A
  \]
- rank,
- characteristic polynomial,
- and eigenvalues.

Similarity is an *equivalence relation* that partitions the set of all \(n \times n\) matrices into classes.

**Example** \(2 \times 2\) matrices
Down the road, you will learn that a representative matrix of each class is called the **Jordan canonical form**.

Ideally, given a matrix, we would like it to be similar to a...diagonal matrix! (Then eigenvalues would be precisely the diagonal entries of the diagonal matrix to which it is similar.)

**Definition** A matrix $A$ is *diagonalizable* if it is similar to a diagonal matrix $D$, i.e.,

$$P^{-1}AP = D$$

for some invertible matrix $P$ and some diagonal matrix $D$

$$\implies AP = PD.$$

**Example** The matrix $A = \begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix}$ is diagonalizable since $A \sim D$ where $D = \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix}$. Check that

$$\begin{bmatrix} 4 & -2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} 2 & 0 \\ 0 & 3 \end{bmatrix},$$

and $P$ is invertible because $\det P \neq 0$.

**Example** We claim that the matrix $A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ is not diagonalizable. Suppose, by way of contradiction (BWOC), that $A$ is diagonalizable, i.e., there exist invertible matrix $P$ and diagonal matrix $D$ such that $P^{-1}AP = D$. Then $A = PDP^{-1}$, and we have

$$0 = A^2 = (PDP^{-1})(PDP^{-1}) = PD^2P^{-1}$$

$$\implies 0 = D^2$$

$$\implies 0 = D$$

$$\implies A = PDP^{-1} = P0P^{-1} = 0 \implies$$

We have arrived at a contradiction, so it must be that $A$ is not diagonalizable.
Questions

• When is $A$ diagonalizable?

• If $A$ is diagonalizable, how do we find matrices $P$ and $D$ such that $P^{-1}AP = D$?

Given matrix $A$, suppose $A$ has eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_n$ with corresponding eigenvectors $\vec{x}_1, \vec{x}_2, \ldots, \vec{x}_n$. Then

\[
\begin{align*}
A\vec{x}_1 &= \lambda_1 \vec{x}_1 \\
A\vec{x}_2 &= \lambda_2 \vec{x}_2 \\
& \vdots \\
A\vec{x}_n &= \lambda_n \vec{x}_n
\end{align*}
\]

\[
A \begin{bmatrix}
\begin{array}{c|c|c|c}
\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n
\end{array}
\end{bmatrix} = \begin{bmatrix}
\begin{array}{c|c|c|c}
\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n
\end{array}
\end{bmatrix} \begin{bmatrix}
\lambda_1 & \lambda_2 & \cdots & \lambda_n
\end{bmatrix}
= \begin{bmatrix}
\begin{array}{c|c|c|c}
\vec{x}_1 & \vec{x}_2 & \cdots & \vec{x}_n
\end{array}
\end{bmatrix} \begin{bmatrix}
\lambda_1 \\
\lambda_2 \\
\vdots \\
\lambda_n
\end{bmatrix}
\implies AP = PD
\]

When is $P$ invertible? \implies eigenvectors of $A$ are linearly independent

**Theorem.** Let $A$ be an $n \times n$ matrix. Then

- $A$ is diagonalizable $\iff$ $A$ has $n$ linearly independent eigenvectors.