Diagonalizability

Encodings, coordinates, and change of basis

Math 40, Introduction to Linear Algebra
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**Definition** A matrix $A$ is **diagonalizable** if it is similar to a diagonal matrix $D$, i.e.,

\[ P^{-1}AP = D \]

for some invertible matrix $P$ and some diagonal matrix $D$.

**Theorem.** Let $A$ be an $n \times n$ matrix. Then

\[ A \text{ is diagonalizable} \iff A \text{ has } n \text{ linearly independent eigenvectors.} \]

If we know that $A$ is diagonalizable, then $P^{-1}AP = D$, and

- columns of $P$ are $n$ linearly independent eigenvectors of $A$,
- and diagonal entries of $D$ are eigenvalues of $A$. 

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Characterization of diagonalizability
Example of diagonalization

**Example** Consider $A = \begin{bmatrix} 3 & 0 & 0 \\ -2 & 4 & 2 \\ -2 & 1 & 5 \end{bmatrix}$.

- Find eigenvalues of $A$.  
  $0 = \det(A - \lambda I)$
  $= (3 - \lambda)((4 - \lambda)(5 - \lambda) - 2)$
  $= (3 - \lambda)(\lambda^2 - 9\lambda + 18)$ \hspace{1cm} $\lambda = 3 \text{ (alg. mult. 2)}$
  $= (3 - \lambda)(\lambda - 3)(\lambda - 6)$ \hspace{1cm} $\lambda = 6$

- Find eigenspaces of $A$.

  Solve $(A - 3I)x = \vec{0}$. $\implies E_3 = \text{span} \begin{bmatrix} 1 \\ 2 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$

  Solve $(A - 6I)x = \vec{0}$. $\implies E_6 = \text{span} \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}$

- Construct matrices $P$ and $D$.

  \[ D = \begin{bmatrix} 3 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 6 \end{bmatrix}, \quad P = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 2 & 0 \\ 0 & 1 & 1 \end{bmatrix}. \]

  Then $AP = PD$ and $P$ is invertible, so $P^{-1}AP = D$ and $A$ is diagonalizable.

Linearly independent eigenvectors

**Theorem.** If $A$ is an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \ldots, \lambda_k$, then the collection of all basis vectors for the eigenspaces $E_{\lambda_1}, E_{\lambda_2}, \ldots, E_{\lambda_k}$ is linearly independent.

Proof uses the fact that distinct eigenvalues have eigenvectors that are linearly independent.

**Corollary.** If $A$ is an $n \times n$ matrix with $n$ distinct eigenvalues, then $A$ is diagonalizable.

In our last example, the $3 \times 3$ matrix $A$ had two distinct eigenvalues:

- $3$ (alg. mult. 2) and $6$ (alg. mult. 1).

We know basis for $E_3 \cup E_6$ basis for $E_6$ is a linearly independent set.

For any eigenvalue, algebraic mult. $\geq$ geometric mult.

Hence, for $A$ to be diagonalizable, we need eigenspace $E_3$ to be two-dimensional.
Full characterization of diagonalizability

**Theorem.** Let $A$ be an $n \times n$ matrix with distinct eigenvalues $\lambda_1, \lambda_2, \ldots, \lambda_k$. Then the following are equivalent:

1. $A$ is diagonalizable.
2. Algebraic and geometric multiplicities are equal for each eigenvalue of $A$.
3. Union of bases of eigenspaces of $A$ has $n$ vectors.

Invertibility and diagonalizability

Which of the following statements are true?

- If $A$ is diagonalizable, then $A$ is invertible.
- If $A$ is invertible, then $A$ is diagonalizable.
- If $A$ is not diagonalizable and not invertible, then $A$ is the matrix of all zeros ($A = 0$).

**All are false!!**
**Differentiation as a linear transformation**

$\mathcal{P}_3 \implies$ set of all polynomials of degree at most 3 with real coefficients
$\mathcal{P}_2 \implies$ set of all polynomials of degree at most 2 with real coefficients

vector spaces

Consider linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ of differentiation:

$$T(p) = p' \quad \text{for polynomials } p \in \mathcal{P}_3.$$  

In other words,

$$T(ax^3 + bx^2 + cx + d) = 3ax^2 + 2bx + c.$$  

Verify that this is a linear transformation.

What is a basis for $\mathcal{P}_3$? \quad What is a basis for $\mathcal{P}_2$?

$B = \{x^3, x^2, x, 1\}$ \quad $D = \{x^2, x, 1\}$

**Encodings and coordinates**

Consider linear transformation $T : \mathcal{P}_3 \to \mathcal{P}_2$ of differentiation:

$$T(p) = p' \quad \text{for polynomials } p \in \mathcal{P}_3.$$  

$B = \{x^3, x^2, x, 1\}$ \quad $D = \{x^2, x, 1\}$

Then

$$p \in \mathcal{P}_3 \implies p = ax^3 + bx^2 + cx + d \implies [p]_B = \begin{bmatrix} a \\ b \\ c \\ d \end{bmatrix} \in \mathbb{R}^4$$

encoding of $p$ with respect to basis $B$,

or coordinates of $p$ with respect to $B$

$$q \in \mathcal{P}_2 \implies q = ax^2 + bx + c \implies [q]_D = \begin{bmatrix} a \\ b \\ c \end{bmatrix} \in \mathbb{R}^3$$

encoding of $q$ with respect to basis $D$
Matrix of the linear transformation

Now that we have a way to encode the polynomials, we consider encoding our linear transformation $T$ using a matrix.

$$T : \mathcal{P}_3 \to \mathcal{P}_2 \implies \left[ \begin{array}{c} 3 \times 4 \text{ matrix} \end{array} \right]$$

$B = \{x^3, x^2, x, 1\}$

To find the matrix of $T$, we ask ourselves what the linear trans. $T$ does to basis vectors.

$D = \{x^2, x, 1\}$

$$\begin{align*}
T(x^3) &= 3x^2 & T(x^2) &= 2x & T(x) &= 1 & T(1) &= 0 \\
[3x^2]_D &= \begin{bmatrix} 3 \\ 0 \\ 0 \\ 0 \end{bmatrix} & [2x]_D &= \begin{bmatrix} 0 \\ 2 \\ 0 \\ 0 \end{bmatrix} & [1]_D &= \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix} & [0]_D &= \begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \end{bmatrix} \\
& & & & & & \\
\left[ T \right]_{D \leftarrow B} &= \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B}
\end{align*}$$

Using the matrix of $T$

How do we find the image of a polynomial $p$ under $T$?

$$\left[ T \right]_{D \leftarrow B} = \begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B}$$

How do we use $T$ to find the derivative of $p$?

**Theorem.**

$$\left[ T(v) \right]_D = \left[ T \right]_{D \leftarrow B} \left[ v \right]_B$$

**Example:** Consider $p(x) = 5x^3 - 3x + 2$. We want to find $T(p(x))$. We have

$$\begin{align*}
\begin{bmatrix} 3 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}_{D \leftarrow B} \begin{bmatrix} 5 \\ 0 \\ -3 \\ 2 \end{bmatrix}_B &= \begin{bmatrix} 15 \\ 0 \\ -3 \end{bmatrix}_D \\
p'(x) &= T(p(x)) \\
&= 15x^2 - 3
\end{align*}$$
Change of basis

In our last example, we chose a basis $B = \{x^3, x^2, x, 1\}$ for the vector space $\mathcal{P}_3 \rightarrow \text{set of all polynomials of degree at most 3 with real coefficients.}$

What if we decided we wanted to use a different basis for this space?

\[ B = \{x^3, x^2, x, 1\} \quad \quad C = \{(x + 1)^3, x^2, x + 1, x - 1\} \]

For example,

\[ [x^3]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}_B \quad \text{and} \quad [x^3]_C = \begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \end{bmatrix}_C \]

Is it possible to find $[x^3]_C$ from $[x^3]_B$ using matrix multiplication?

**YES!!**
Change of basis

What is the relationship between encodings of polynomials with respect to $B$ and those with respect to $C$?

\[ B = \{ x^3, x^2, x, 1 \} \quad \quad C = \{ (x + 1)^3, x^2, x + 1, x - 1 \} \]

For example,

\[ [x^3]_B = \begin{bmatrix} 1 \\ 0 \\ 0 \\ 0 \end{bmatrix}_B \quad \text{and} \quad [x^3]_C = \begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \end{bmatrix}_C \]

To find the matrix of $I$, we ask ourselves what the linear trans. $I$ does to basis vectors.

\[ \begin{align*}
I(x^3) &= x^3 \\
I(x^2) &= x^2 \\
I(x) &= x \\
I(1) &= 1
\end{align*} \]

\begin{align*}
[x^3]_C &= \begin{bmatrix} 1 \\ -3 \\ -2 \\ -1 \end{bmatrix}_C \\
[x^2]_C &= \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}_C \\
[x]_C &= \begin{bmatrix} 0 \\ \frac{1}{2} \\ \frac{1}{2} \end{bmatrix}_C \\
[1]_C &= \begin{bmatrix} 0 \\ \frac{1}{2} \\ -\frac{1}{2} \end{bmatrix}_C
\end{align*} \]

Computing change-of-basis matrix

\[ B = \{ x^3, x^2, x, 1 \} \quad \quad C = \{ (x + 1)^3, x^2, x + 1, x - 1 \} \]

\[
[ I ]_{c\rightarrow b} = \begin{bmatrix}
1 & 0 & 0 & 0 \\
-3 & 1 & 0 & 0 \\
-2 & 0 & \frac{1}{2} & \frac{1}{2} \\
-1 & 0 & \frac{1}{2} & -\frac{1}{2}
\end{bmatrix}_{c\rightarrow b}
\]