The relative canonical sheaf and base change

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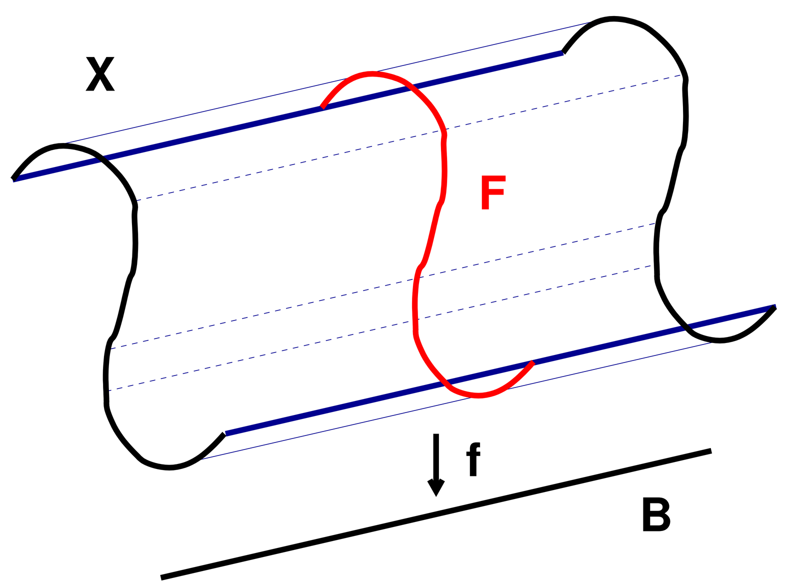
Question

What do we know about the behavior of the relative canonical sheaf of flat families under base change?



i.e. do we have:

$$\omega_{X/B}|_F \cong \omega_F \quad ? \tag{1}$$

It is certainly true for families of smooth varieties, since then

$$\omega_{X/B} \cong \det \Omega_{X/B} \qquad \omega_F \cong \det \Omega_F$$

and (1) follows from compatibility of the sheaf of relative Kähler differentials with base change. However we are interested in the following.

Precise question

Does (1) hold for flat families of projective normal varieties, or more generally for flat families of projective, S_2, G_1 schemes (of finite type over $\mathbb{C}$)?

Motivation

Main motivation: compact moduli of higher dimensional varieties.

Understanding the base change properties of relative canonical sheaves is a vital point in higher dimensional moduli theory. The existence and the prescription of certain compatibilities determine the reduced and the non-reduced structure of the moduli space. For example given a flat family $f : X \to B$ of S_2, G_1 schemes, for f to be a *stable family*:

- We always require that

$$\exists n : \omega_{X/B}^{[n]} \text{ is a line bundle} \tag{2}$$

  One consequence of this is that $\omega_{X/B}^{[n]}|_{X_b} \cong \omega_{X_b}^{[n]}$ for every $b \in B$ (Lemma 2.6. of [HK04]). Without (2) the numerical invariants (e.g. $K_{X_b}^{\dim X_b}$) would not be constant in stable families.

- Sometimes we also require the condition

$$\omega_{X/B}^{[n]}|_{X_b} \cong \omega_{X_b}^{[n]} \qquad (\forall n) \tag{3}$$

  This is the so called *Kollár's condition*. It governs the nilpotent structure of the moduli space.

As a special case, understanding (3) for $n = 1$ is an important issue in higher dimensional moduli theory. The state of the art is as follows. Recently in [KK09]:

- it is proven that (1) holds if the fibers are Du Bois
- it is proven that log canonical varieties are Du Bois
- it is announced that semi-log canonical schemes are Du Bois

Also it is shown in [C00], Theorem 3.5.1 that (1) holds whenever the fibers are Cohen-Macaulay.

Known facts

(1) holds whenever fibers are either Du Bois (in particular semi-log canonical) or Cohen-Macaulay.

The question is if these results are sharp.

Result

We answer the question posed above.

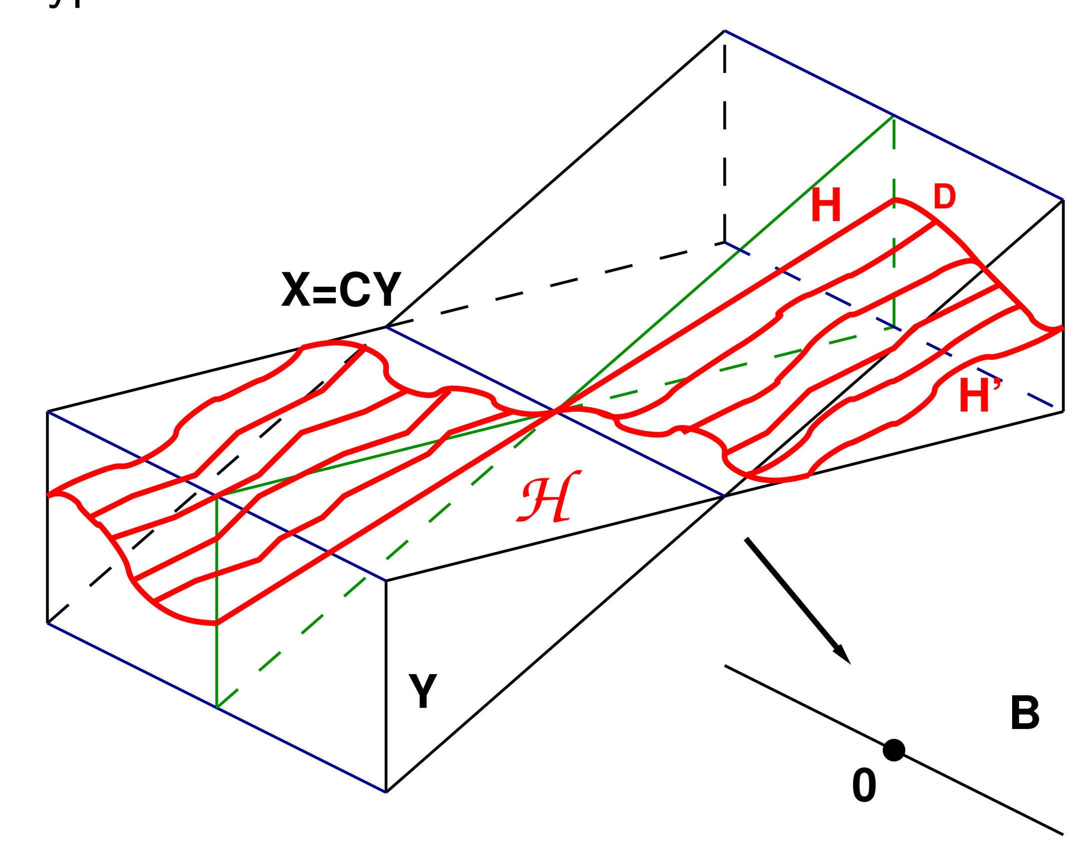
Theorem

(1) does not hold generally for flat families of S_2, G_1 schemes. More precisely we present families $f : X \to B$ whose fibers are normal and S_{n-1} or normal and $\mathbb{Q}$-Gorenstein and do not satisfy (1). Our examples have also $\mathbb{Q}$-line bundle $\omega_{X/B}$.

Construction

To prove our theorem, we need a construction that exhibits examples, where the relative canonical sheaf is not compatible with base change. We start with a projective variety Y with mild singularities satisfying certain cohomological criteria, explained later. Then we consider the following objects:

- the projective cone $X = CY$ over Y
- a general hypersurface H' in X of degree $d \gg 0$ (which by the generality hypothesis avoids the vertex P of X)
- a general hypersurface D of degree d in Y.
- the projective cone H over D, viewed as a hypersurface in X
- the family $\mathcal{H}$ of hypersurfaces in X, corresponding to the line joining H and H' in the moduli of such hypersurfaces



Idea

By [KM98], Theorem 5.69., ω_H is S_2. So, to prove, that $\omega_{\mathcal{H}/B}|_H \ncong \omega_H$, it is enough to prove that $\omega_{\mathcal{H}/B}|_H$ is not S_2. Using that ω_X is S_2 and $\mathcal{H}$ is relatively G_1 (if Y is G_1, this is satisfied too) we get:

$$\omega_{\mathcal{H}/B} \cong \omega_{X\times B/B}(\mathcal{H})|_{\mathcal{H}}$$

Hence it is enough to show that $(\omega_{X\times B/B}(\mathcal{H})|_{\mathcal{H}})|_H$ is not S_2. However there is an isomorphism

$$(\omega_{X\times B/B}(\mathcal{H})|_{\mathcal{H}})|_H \cong (\omega_{X\times B/B}(\mathcal{H})|_{X\times\{0\}})|_H \cong$$
$$\cong \omega_{X\times\{0\}}(H)|_H$$

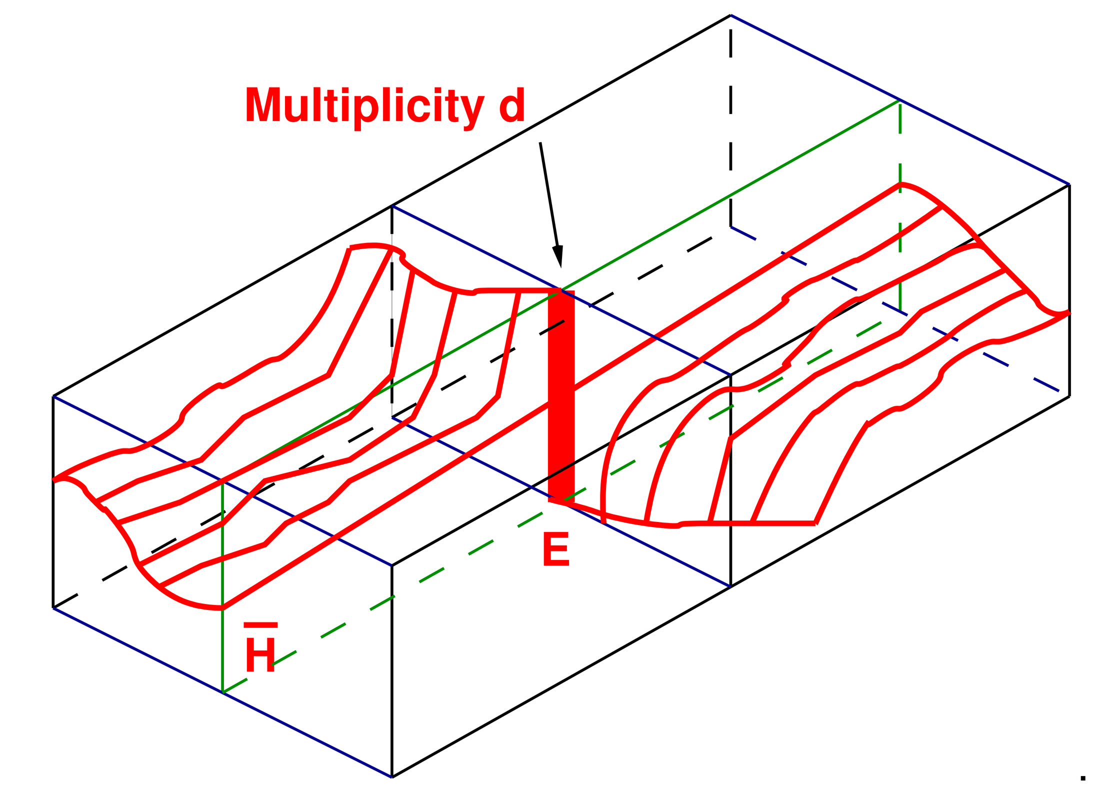
The (outline of the) proof

From the above idea we see, that to prove our theorem, it is enough to exhibit a Y, such that $\omega_X|_H$ is S_1 but not S_2. By some local cohomology techniques, for $d \gg 0$ this is equivalent to ω_X being S_2 but not S_3. This can be obtained by using the following lemma.

Lemma If Y is Cohen-Macaulay and $2 \le d \le \dim X$ then ω_X is S_d if and only if $H^i(\omega_Y(n)) = 0$ for all $0 < i < d - 1$ and $n \in \mathbb{Z}$.

Stable reduction

In the view of [KK09], the base change of the relative canonical sheaf fails, because H is not the right limit of the family $\mathcal{H}|_{B\setminus\{0\}}$. If we used the stable limit instead of H, then there would be nice base change behavior. Here we give a picture of what the stable limit is, and how it differs to our "wrong" limit. We take our original construction. First, we blow up the vertex of X:



The exceptional divisor of the strict transform of $\mathcal{H}$ is going to be then E which is isomorphic to Y with multiplicity d. The other component of the central fiber is going to be the $\bar{H}$, the strict transform of H. Semi-stable reduction then replaces E by a cyclic covering Z of Y of degree d branched over D. Finally, passage to the relative canonical model contracts $\bar{H}$, and the stable limit is going to be Z.

References

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