# Modeling Flight over a Spherical Earth

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May 9, 2008

## 1 Introduction

Deriving the equations of motion of a point-mass projectile moving in a vacuum solely under the influence gravity is one of the first things we learn regarding the physics of flight. This model of flight is of course greatly simplified and is not an accurate enough description of the motion of objects for almost all practical modern-day settings. We wish to expand the model of flight to be able to more accurately model a ballistic missile whose trajectory is flown in the immediate neighborhood of the Earth. Beyond simply modeling a rocket that has variable thrust and mass, we will take into account that the surface over which the rocket is traveling, the earth, is spherical in shape, rotates around its axis, and that the strength of gravity varies depending on the altitude of the missile. We will examine a more specific case of flight in that we will study a class of paths which are flown in a great-circle plane. A great circle is a circle on the surface of a sphere, in this case the Earth, that has the same circumference as the sphere. The corresponding great-circle plane is the plane which contains this great circle. The great circle we will be choosing will be perpendicular to the equator, or along a fundamental parallel, which will be defined in more detail later.

## 2 The Model

If we assume that the rocket is a solid body (i.e. there is no fluid within the rocket that could affect its motion) and take into account the Coriolis acceleration acting on the rocket that results from the rotation of the Earth, the general dynamical equation that describes the absolute motion of the rocket is

$$\mathbf{T} + \mathbf{A} + m\mathbf{g} = m\left[\frac{d\mathbf{V}}{dt} + 2\boldsymbol{\omega}_e \times \mathbf{V}\right] \tag{1}$$

where $\mathbf{T}$ is the thrust produced by the rocket, $\mathbf{A}$ is the total aerodynamic force, $m$ is the mass of the rocket, $\boldsymbol{\omega}_e$ is angular velocity of the Earth with respect to the Fixed Stars, and $\mathbf{g}$ is the acceleration of gravity. Also, $\mathbf{V}$ is the velocity of the rocket with respect to the Earth and is given by

$$\mathbf{V} = \frac{d\mathbf{EO}}{dt} \tag{2}$$

where $\mathbf{EO}$ is the vector joining point $E$ on the surface of the Earth with point $O$ on the rocket, as shown in Figure 1.

We wish to find the scalar equations associated with this vectorial relationship so that we can find the location of the rocket on the sphere at a specific time under given initial conditions. To do this we must define several relevant coordinate systems and then establish the relationships between position and motion in one system with respect to another. We do this because many of the forces involved are produced by the rocket and are easiest to implement in the rocket's frame of reference, but ultimately we want to find the rocket's position in the Earth's frame of reference, so we have to be able to transform between the two systems.

## 2.1 Coordinate Systems

We employ four different coordinate systems in our investigation; the Earth axes system $Ex_ey_ez_e$, the curvilinear ground system $EXYZ$, the local horizon system $Ox_hy_hz_h$, and the body axes system $Ox_by_bz_b$.

The Earth axes system is a Cartesian reference frame which is rigidly attached to the Earth. Its origin E is a point on the Earth's surface; the $z_e$-axis is vertical and positive downward; the $x_e$-axis and the $y_e$-axis are tangent to the Earth's surface and are directed such that the system is right-handed. The the great circle tangent to the $y_e$-axis is called the fundamental meridian, while the great circle tangent to the $x_e$-axis, our desired path, is called the fundamental parallel.

The curvilinear ground system is an orthogonal reference frame which is fixed to the Earth. Its origin $E$ is a point on the Earth's surface; the $X$-coordinate is measured from $E$ on the fundamental parallel; the $Y$-coordinate is measured from $E$ on the fundamental meridian; and the $Z$-coordinate is measured radially from $E$. Furthermore, the positive senses for $X$, $Y$, $Z$ are consistent with the positive sense for the Earth axes. These three coordinate are sufficient to determine the position of the rocket with respect to the Earth.

The local horizon system is a Cartesian reference frame that has its origin at the instantaneous position of the aircraft $O$; the $z_h$-axis is vertical and positive downward; the $x_h$-axis and the $y_h$-axis are contained in the plane tangent to the spherical surface passing through the aircraft and are such that $Ox_hy_hz_h$ is right-handed. This sphere surface, which has the same center as the Earth sphere, is called the local horizon.

The body axes system also has its origin at $O$; the $x_b$-axis is tangent to the flight path and is positive forward; the $z_b$-axis is perpendicular to the $x_b$-axis, contained in the plane of symmetry of the rocket, and positive downward for the normal flight attitude of the aircraft; the $y_b$-axis is directed such that $Ox_by_bz_b$ is right-handed.

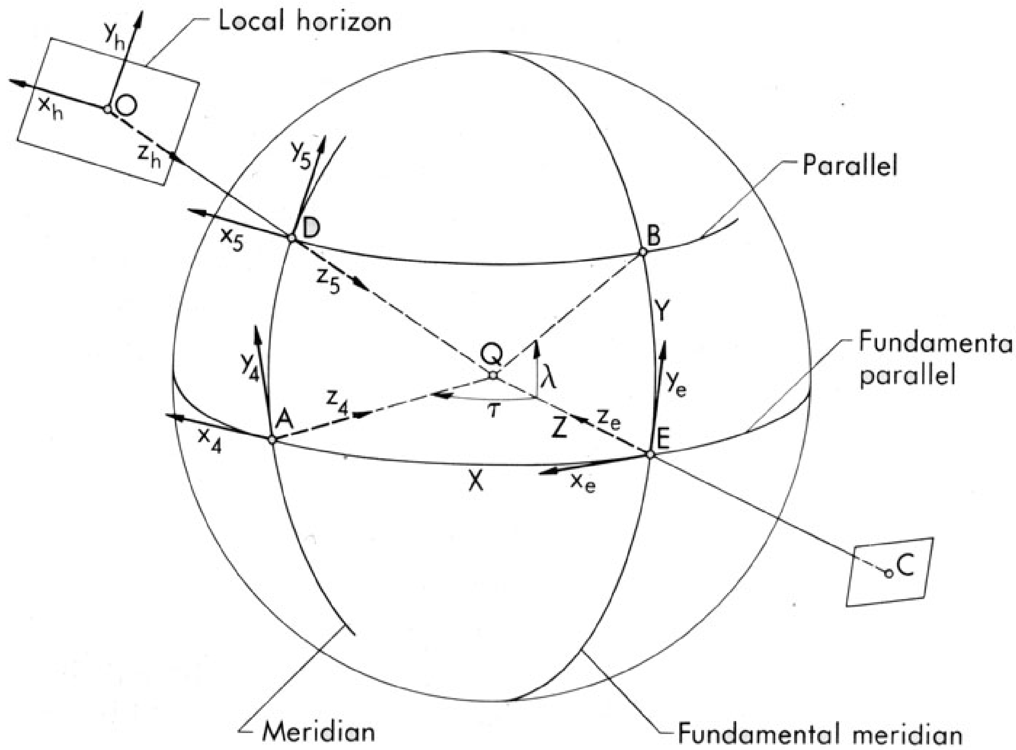



Figure 1. Depicts the Earth and local horizon axes, as well as the intermediate axes used to transform between the two. Also shows the curvilinear ground system and labels all points and angles as defined in the previous and next sections. The rocket travels above the labeled fundamental parallel, which is constructed based on the chosen location of E, the origin of the Earth system.

## 2.2 Angular Relationships

We wish to be able to convert from the body axes system to the Earth axes system and vice versa since most of the forces act on either the rocket or the Earth. We will therefore investigate the angular relationships between the local horizon-Earth axes and body axes-local horizon pairs.

The orientation of the local horizon with respect to the Earth axes can be described in terms of two angular parameters, the longitude $\tau$ and the latitude $\lambda$. Looking at Figure 1 we can see that the necessary rotations to transform from one system to the other are easily understood if two intermediate coordinate systems are introduced. The system $Ax_4y_4z_4$ is obtained from the Earth axes system by means of a rotation $\tau$ around the $y_e$-axis plus a translation; the system $Dx_5y_5z_5$ is obtained from $Ax_4y_4z_4$ by means of a rotation $\lambda$ around the $x_4$-axis plus a translation; finally, the local horizon system is such that its axes and the corresponding axes of the system $Dx_5y_5z_5$ are parallel and have the same positive sense. Using the arc length formula, from Figure 1 it is clear that the curvilinear coordinates and the angles of rotation are related by $\mathrm{X} = \mathrm{r_o}\tau$ and $\mathrm{Y} = \mathrm{r_o}\lambda$, where $\mathrm{r_o}$ is the radius of the Earth. Using matrix-vector notation we can therefore represent the total rotation in the form

$$\begin{bmatrix} \mathrm{i}_h \\ \mathrm{j}_h \\ \mathrm{k}_h \end{bmatrix} = \begin{bmatrix} \cos(\mathrm{X}/r_o) & 0 & \sin(\mathrm{X}/r_o) \\ -\sin(\mathrm{X}/r_o)\sin(\mathrm{Y}/r_o) & \cos(\mathrm{Y}/r_o) & \cos(\mathrm{X}/\mathrm{r_o})\sin(\mathrm{X}/r_o) \\ -\sin(\mathrm{X}/r_o)\cos(\mathrm{Y}/r_o) & -\sin(\mathrm{Y}/r_o) & \cos(\mathrm{X}/\mathrm{r_o})\cos(\mathrm{Y}/r_o) \end{bmatrix} \begin{bmatrix} \mathrm{i}_e \\ \mathrm{j}_e \\ \mathrm{k}_e \end{bmatrix}. \tag{3}$$

The orientation of the body axes with respect to the local horizon can be described in terms of two angular parameters, the velocity yaw $\chi$ and the velocity pitch $\gamma$. Using a similar process as for the local horizon-Earth axes transformation, we introduce an intermediate coordinate system such that the system $Ox_1y_1z_1$ is obtained from the local horizon by means of a rotation $\chi$ around the $z_h$-axis and the wind axes system is obtained from $Ox_1y_1z_1$ by means of a rotation $\gamma$ around the $y_1$-axis. The velocity yaw essentially measures angular deviation of the $x_b$ body axis from the $x_h$ local horizon axis, which is contained in the plane normal to the local horizon. The velocity pitch measures the angular deviation of the $y_b$ body axis from the $y_h$ local horizon axis in the plane containing the $x_h$ and $z_h$ axes. Therefore the total rotation takes the form

$$\begin{bmatrix} \mathrm{i}_w \\ \mathrm{j}_w \\ \mathrm{k}_w \end{bmatrix} = \begin{bmatrix} \cos\gamma\cos\chi & \cos\gamma\sin\chi & -\sin\gamma \\ -\sin\chi & \cos\chi & 0 \\ \sin\gamma\cos\chi & \sin\gamma\sin\chi & \cos\gamma \end{bmatrix} \begin{bmatrix} \mathrm{i}_h \\ \mathrm{j}_h \\ \mathrm{k}_h \end{bmatrix}. \tag{4}$$

## 2.3 Angular Velocity

Because we are dealing with a rotating Earth, we must take into account the forces acting on the rocket that come from the angular velocity of the Earth. We therefore need to determine how the angular velocity is transformed when going from the Earth axes system to the local horizon system.

When going from the Earth axes to the local horizon system, we rotated along the $y_e$ and $x_4$ axes. The infinitesimal angular displacement of the local horizon with respect to the Earth is therefore given by

$$d\mathbf{\Omega}_h = d\lambda\mathbf{i}_4 - d\tau\mathbf{j}_e.$$

Consequently, the angular velocity of the local horizon with respect to the Earth becomes

$$\boldsymbol{\omega}_h = \frac{d\mathbf{\Omega}_h}{dt} = \dot{\lambda}\mathbf{i}_4 - \dot{\tau}\mathbf{j}_e$$

which, in consideration of Eq. (3) and the transformation matrices to the intermediate coordinate systems, can be rewritten in the form

$$\boldsymbol{\omega}_h = \frac{\dot{Y}}{r_o}\mathbf{i}_h - \frac{\dot{X}}{r_o}\cos(\mathrm{Y}/\mathrm{r_o})\mathbf{j}_h + \frac{\dot{X}}{r_o}\sin(\mathrm{Y}/\mathrm{r_o})\,\mathbf{k}_h. \tag{5}$$

We also need to know the evolutory velocity, that is, the angular velocity of the body axes with respect to the Earth axes, which we will represent as

$$\boldsymbol{\omega}_b = p_w\mathbf{i}_b + \mathrm{q}_b\mathbf{j}_b + \mathrm{r}_b\mathbf{k}_{b.} \tag{6}$$

## 2.4 Kinematic Relationships

We must now derive the scalar relationships corresponding to the vectorial equation (2). Because the velocity of the rocket is collinear with the $x_b$-axis, we can write the velocity as

$$\mathbf{V} = V\mathbf{i}_b = V(\cos\gamma\cos\chi\,\mathbf{i}_h + \cos\gamma\sin\chi\,\mathbf{j}_h - \sin\gamma\mathbf{k}_h) \tag{7}$$

after using the transformation matrix in Eq. (4). We can then rewrite the vector joining the origin of the Earth axes system with the aircraft as

$$\mathbf{EO} = \mathbf{EQ} - (r_o + h)\mathbf{k}_h$$

where **EQ** is a vector rigidly attached to the Earth and $h$ is the altitude of the rocket above sea level as shown in Figure 1. If we then take the time derivate of this equation as is required for Eq. (2) we get

$$\frac{d\mathbf{EO}}{dt} = -\dot{h}\mathbf{k}_h - (r_o + h)\frac{d\mathbf{k}_h}{dt} \tag{8}$$

where , because of Poisson's formulas, the time derivative of the unit vector perpendicular to the local horizon is given by

$$\frac{d\mathbf{k}_h}{dt} = \boldsymbol{\omega}_h \times \mathbf{k}_h = -\frac{\dot{X}}{r_o}\cos(\mathrm{Y}/\mathrm{r_o})\mathbf{i}_h - \frac{\dot{Y}}{r_o}\mathbf{j}_h. \tag{9}$$

As a final step, we combine Eqs. (2), (7), (8), and (9) to get the following kinematic relationships:

$$\dot{X} = V\frac{r_o}{r_o + h}\frac{\cos\gamma\cos\chi}{\cos(Y/r_o)} \tag{10}$$

$$\dot{Y} = V\frac{r_o}{r_o + h}\cos\gamma\cos\chi$$

$$\dot{h} = V\sin\gamma$$

## 2.5 Dynamic Relationships

We will now derive the scalar form of the vectorial equation (1) using a process similar to that of the previous section, in which we determined the components of each vector on the body axes.

We first introduce the thrust angle of attack $\epsilon$, which determines the angle of rotation in the $x_b$ and $z_b$ plane to which the body axes system must be subjected in order to turn the $x_b$-axis in a direction parallel to the thrust. Consequently, the thrust becomes

$$\mathbf{T} = T[\cos\epsilon\,\mathbf{i}_b - \sin\epsilon\,\mathbf{k}_b]. \tag{11}$$

We then write the aerodynamic force in terms of its components on the body axes as

$$\mathbf{A} = -(D\mathbf{i}_b + \mathrm{Q}\mathbf{j}_b + \mathrm{L}\mathbf{k}_b) \tag{12}$$

where *D* is the drag, *Q* the side force, and *L* the lift. We then note that the acceleration of gravity has the same direction as the $z_h$-axis and can write it as

(13)

$$\mathbf{g} = g[-\sin\gamma\,\mathbf{i}_b + \cos\gamma\,\mathbf{k}_b]$$

We then take the time derivative of Eq. (7) to find the acceleration of the aircraft relative to the Earth and get

$$\frac{d\mathbf{V}}{dt} = \dot{V}\mathbf{i}_b + V\frac{d\mathbf{i}_b}{dt} = \dot{V}\mathbf{i}_b + V(\boldsymbol{\omega}_b \times \mathbf{i}_b) = \dot{V}\mathbf{i}_b + \mathrm{Vr}_b\mathbf{j}_b - \mathrm{Vq}_b\mathbf{k}_b \tag{14}$$

after we use Poisson's formulas to find the time rate of change of the unit vector tangent to the flight path in conjunction with Eq. (6) and substitute the result into the equation.

We must now examine the Coriolis acceleration term, which can be written as follows

$$\mathbf{a}_c = 2\boldsymbol{\omega}_e \times \mathbf{V} = 2V(\mathrm{r}_{eb}\mathbf{j}_b - \mathrm{q}_{eb}\mathbf{k}_b) \tag{15}$$

where $\mathrm{p}_{eb}$, $\mathrm{r}_{eb}$, and $\mathrm{q}_{eb}$ denote the components of the angular velocity of the Earth on the body axes.

We then apply the law of variation of the acceleration of gravity with the altitude to get

$$g = g_o\left(\frac{r_o}{r_o + h}\right)^2. \tag{16}$$

If we then combine Eqs. (1) and (11)-(16), the following scalar equations are derived:

$$T\cos\epsilon - D - mg_o\left(\frac{r_o}{r_o + h}\right)^2 \sin\gamma - m\dot{V} = 0 \tag{17}$$

$$T\sin\epsilon + L - mg_o\left(\frac{r_o}{r_o + h}\right)^2 \cos\gamma - mV(q_b + 2q_{eb}) = 0$$

$$-Q - mV(r_b + 2r_{eb}) = 0$$

## 2.6 Governing Equations

We have derived general equations for the motion of the rocket over the Earth, but we wish to examine the case where the rocket travels solely in a great-circle plane. The great circle we will take under consideration is the fundamental parallel, the particular motion of which is expressed mathematically by setting $Y = 0$ and $\chi = 0$ at all time instants. Under this condition, the components of the evolutory velocity on the body axes simplify to

$$p_b = 0, \qquad q_b = \dot{\gamma} - \frac{\dot{X}}{r_o}, \qquad r_b = 0. \tag{18}$$

As a final result, the kinematic and dynamic relationships describing the motion of an aircraft in a great-circle plane are given by the reduced forms of Eqs. (10) and (17) with Eq. (18) to get

$$\dot{X} = V\frac{r_o}{r_o + h}\cos\gamma$$

$$\dot{h} = V \sin\gamma$$

$$T\cos\epsilon - D - mg_o\left(\frac{r_o}{r_o + h}\right)^2 \sin\gamma = m\dot{V}$$

$$T\sin\epsilon + L - mg_o\left(\frac{r_o}{r_o + h}\right)^2 \cos\gamma = mV[\dot{\gamma} - \frac{V\cos\gamma}{r_o + h} + 2q_{eb}]$$

In this system of equations there is one independent variable, the time, and four dependent variables, $X, h, V$, and $\gamma$, which we have to solve for. The values for $r_o$ and $q_{eb}$ can be found in tables, leaving the user to specify values for $T, D, L, m$, and $\epsilon$, as well as the initial values for the four dependent variables. Figure 2 shows the directions of the relevant parameters in the great-circle plane in which the rocket is traveling.

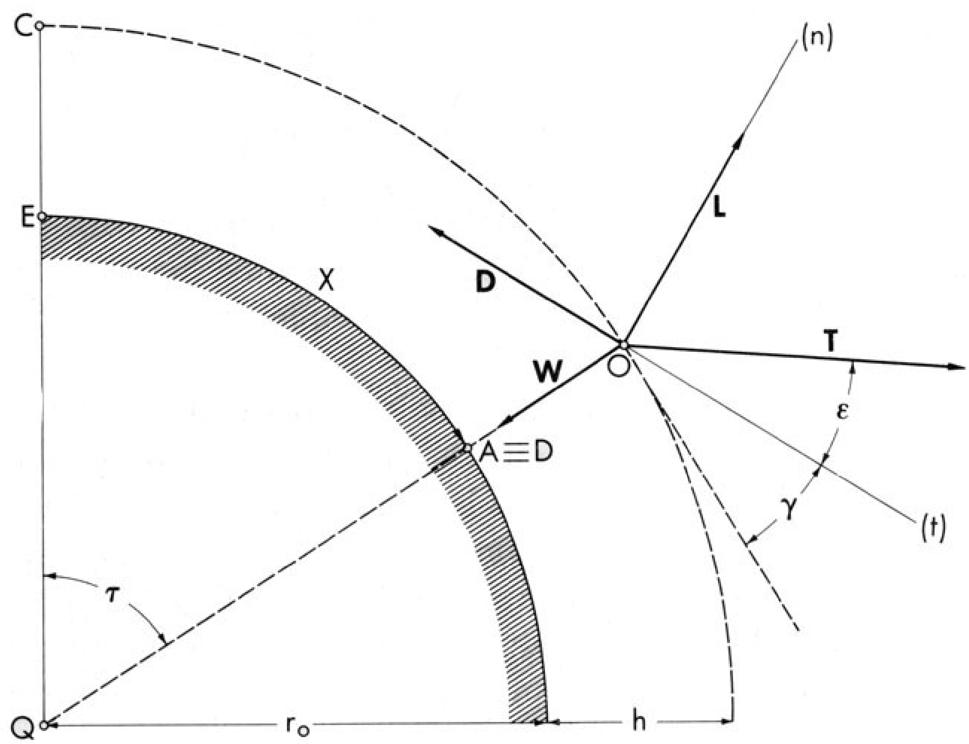
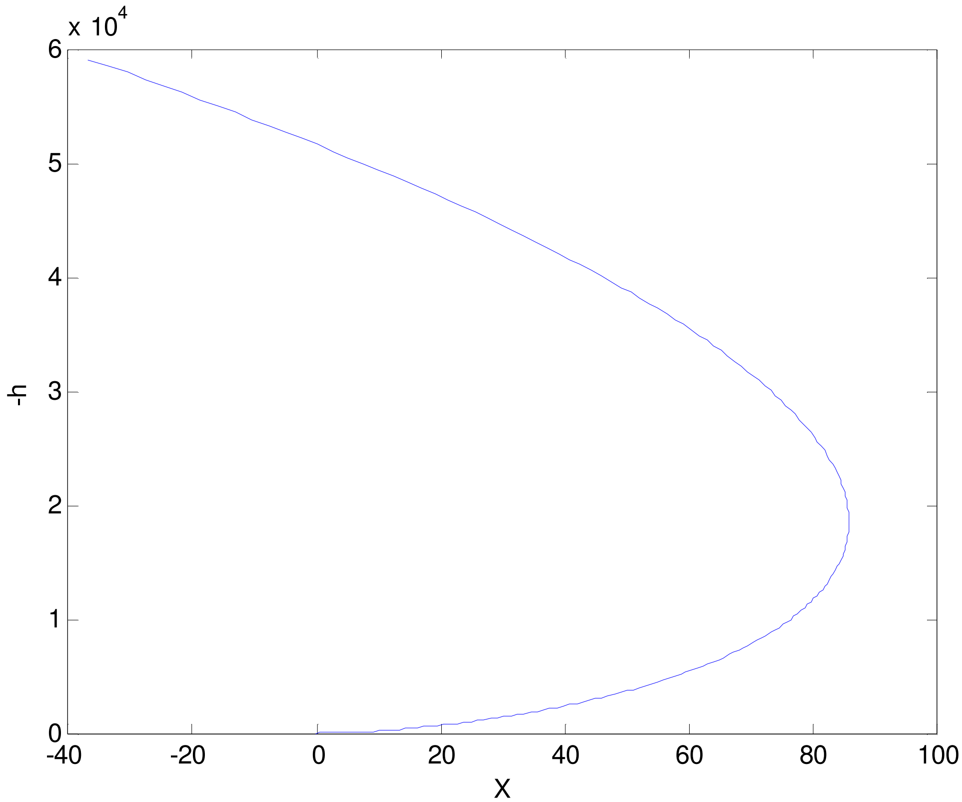



Figure 2. Shows the great-circle plane in which the rocket travels as a cross-section of the Earth. The instantaneous direction the rocket is traveling in is labeled as (t). All other angles and forces are labeled as described in the previous sections.

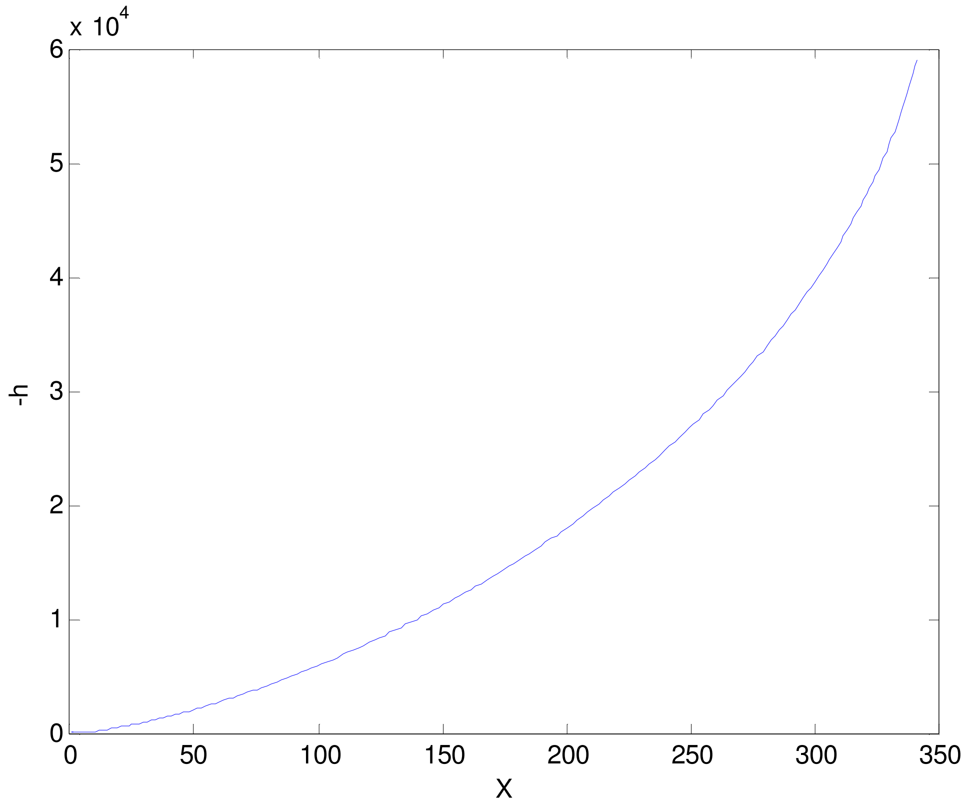
# 3 Results

We will study the trajectory of the rocket for various values of initial velocity pitch angle ($\gamma_o$), thrust angle of attack ($\epsilon$), drag ($D$), lift ($L$), mass ($m$), initial velocity ($V_o$),and thrust ($T$). We will use a value of 6367442.5m for $r_o$ and a value of $7.2921159 \times 10^{-5}\ {rad}/{s}$ for $q_{eb}$, both values of which can be easily looked up.

We will track the motion for 100s and then plot the parametric curve with horizontal distance ($X$) on the x-axis and vertical distance ($-h$) on the y-axis. Because of the way our coordinate system is oriented, we will plot the negative of the vertical distance since the positive direction for $h$ is directed towards the Earth, and we wish to see the height away from the Earth. We will begin with the following initial values: $m = 1000kg$, $X_o = 0m$, $h_o = 0$, $\epsilon = 0$, $L = 0$, $D = 0$, $T = 2000N$, $\gamma_o = {}^{\pi}/_{4}$, and $V_o = 1\,{}^{m}/_{s}$. We obtain the following graph.

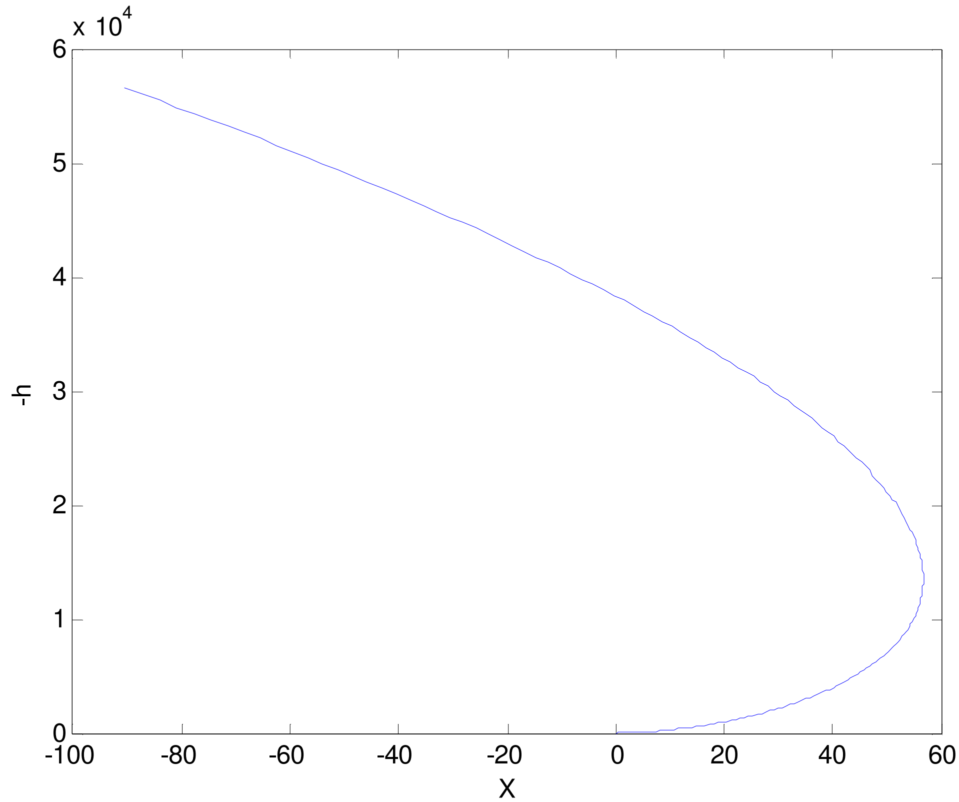



We can see that with the thrust always pointing in the direction of motion of the rocket and no drag or lift to influence the path, the rocket will travel in the direction first oriented and then will loop back on itself. It travels far higher than it does longitudinally and doesn't continue in the direction desired so the initial settings are very impractical for real world use. We will now modify only the value of $\epsilon$ to see the effects of having the thrust point in a different direction than the orientation of the rocket. We will set $\epsilon = {}^{\pi}/_{100}$. This is essentially pointing the thrust away from the Earth by a very small amount, as depicted in Figure 2. The following graph is obtained.

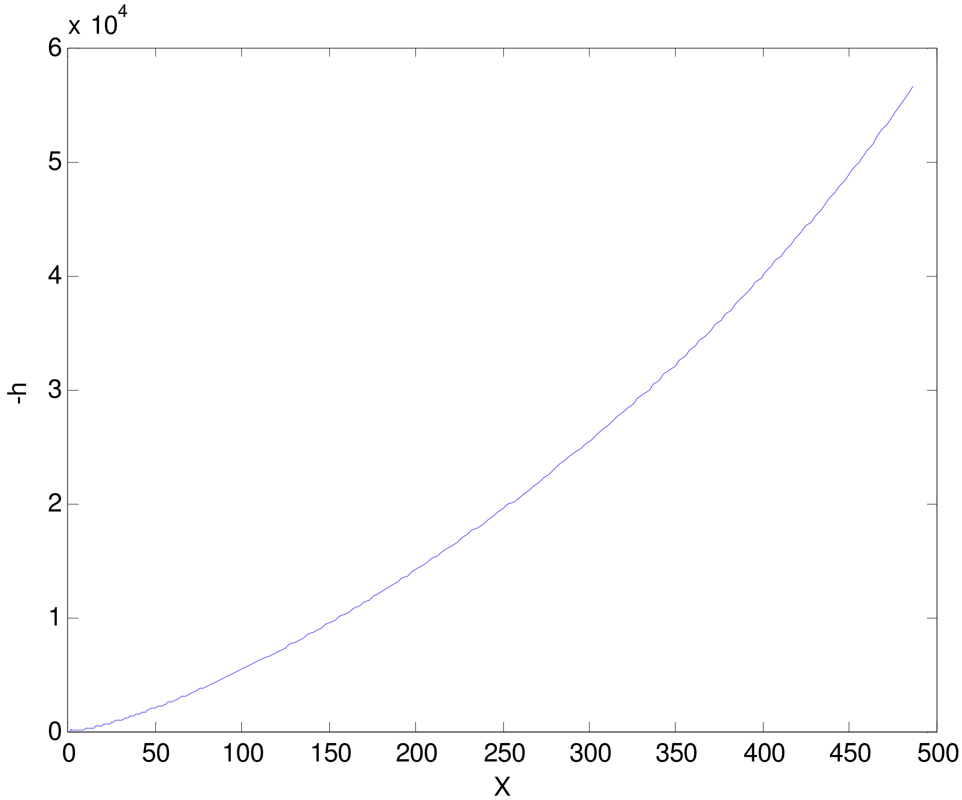



We can see that having the thrust point towards the Earth corrected the problem of having the rocket loop back on its path for this time interval and also resulted in the rocket traveling farther in the longitudinal direction. Increasing the time interval however shows that the rocket does indeed loop back on itself, though further increasing the value of $\epsilon$ corrects this.

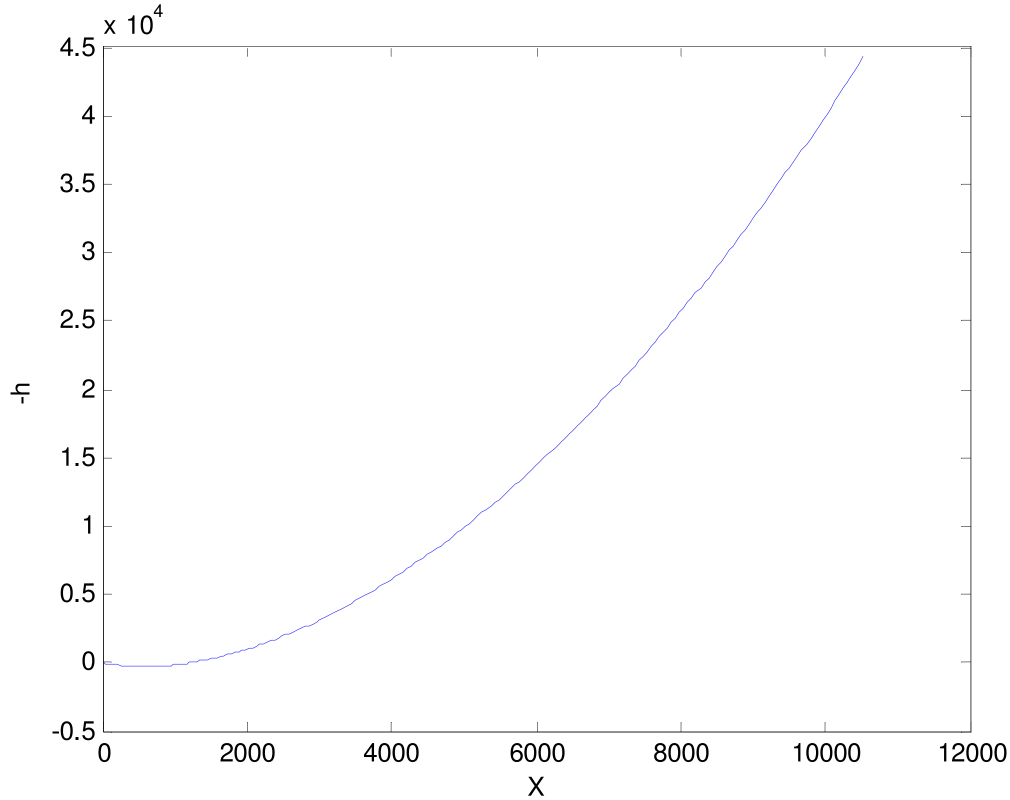
We will now vary the amount of drag placed on the rocket. Leaving $\epsilon = 0$, we set $D = 500N$ and obtain the following graph.



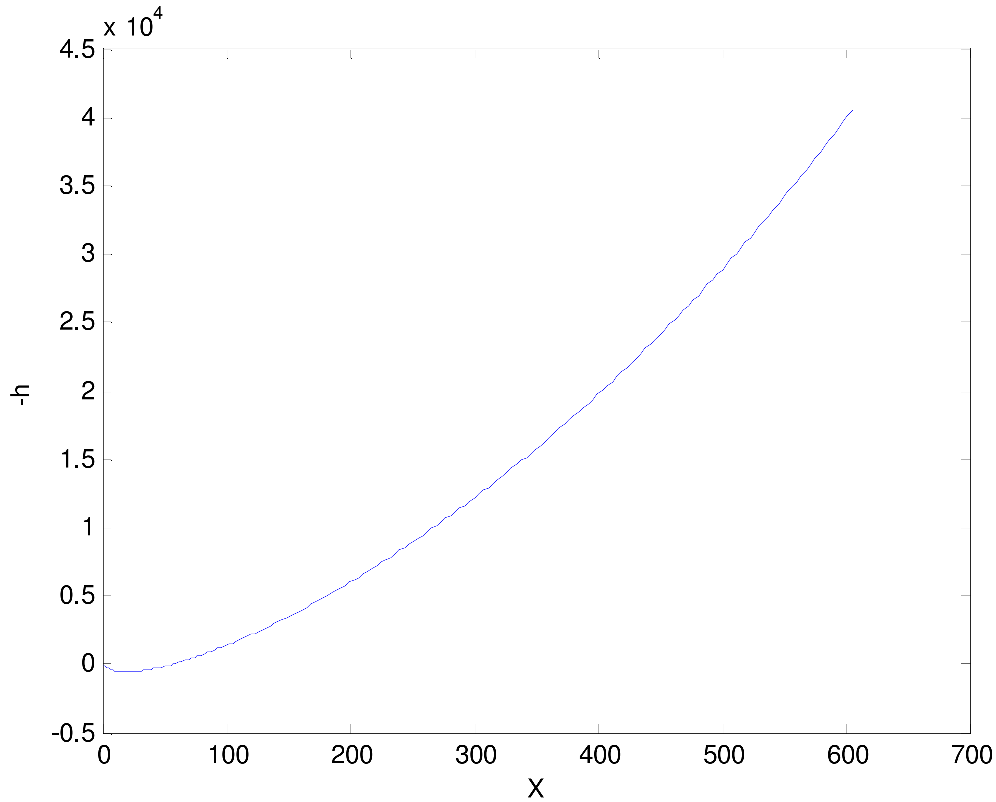

We can see that increasing the drag merely decreases the speed of the rocket, as expected. Compared to there being no drag, the rocket simply doesn't go as far before reversing its course. Increasing the amount of drag amplifies this effect. If we keep this amount of drag and then set $L = 100N$, we get the following graph.




We can see that adding even a small amount of lift greatly straightens out the trajectory and allows the rocket to go farther. We next increase the initial velocity $V_o$ to $100\,{}^{m}/_{s}$ and obtain the following graph.




Increasing the initial velocity greatly increases the range of the rocket, increasing it by a factor of about 24 over having no initial velocity. Increasing the initial velocity even more increases the range by equally large factors. The last parameter to test is the initial velocity pitch angle $\gamma_o$, which we first increase to $^{89\pi}/_{180}$ in order to see the effects of velocity pitch that is nearly straight up vertically. The result is intuitive; the rocket doesn't travel as far longitudinally.




Correspondingly, lower the velocity pitch angle from $45°$ causes the rocket to go a bit farther.

# 4 Conclusion

We have only studied the most general possibilities for the various parameters involved. We have kept their values as constants when in fact they should vary highly based on the position and speed of the rocket. More accurate models for drag and lift are in themselves very complicated problems and beyond the scope of this project. We also used a very simple model for thrust and mass. In a typical rocket, thrust would only be available for so long as there is fuel, and as the fuel is consumed, the mass of the rocket would decrease. Also, the thrust angle of attack $\epsilon$ is a parameter that can be controlled during flight and is changed to direct the rocket towards a particular target. We have also not observed any attempts to land the rocket. This is due to our particular combinations of parameters and mostly because we use a constant thrust. In a real rocket once the thrust stopped a combination of drag and gravity would cause the rocket to fall back to the Earth.

Despite these limitations however, we have verified the general behavior of the rocket in response to major changes in parameter values. A more detailed verification however would need to include more detailed models of lift, drag, thrust, mass, and thrust angle of attack if we wish to model a real rocket's trajectory.

# 5 References

Miele, A., *Flight Mechanics: Theory of Flight Paths*, Addison-Wesley, 1962; pp. 35-68.