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Strings, Chains, and Ropes*

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Abstract. Following Antman [*Amer. Math. Mon.*, 87 (1980), pp. 359–370], we advocate a more physically realistic and systematic derivation of the wave equation suitable for a typical undergraduate course in partial differential equations. To demonstrate the utility of this derivation, three applications that follow naturally are described: strings, hanging chains, and jump ropes.

Key words. partial differential equations, wave equation, elastic one-dimensional continua, hanging chain, jump rope

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1. Introduction.

The derivation of the wave equation is an important moment in students' understanding of partial differential equations (PDEs), as it is often the first PDE that students learn how to derive and solve. In 1980, Stuart Antman wrote a paper advocating an "honest derivation of the classical equations of motion" for strings [1]:

> Many elementary books on partial differential equations ostensibly show that the wave equation in one spatial dimension describes the small transverse vibrations of an elastic string. Of these books I know of but one, namely [11], whose development of the wave equation does not invoke such unjustified simplifications as the assumption that the motion of each particle of the string is confined to a plane perpendicular to the line joining the ends of the string.

A survey of popular PDE textbooks in use today reveals a similar situation. To this author's knowledge, Pinsky's book [10] is the only popular PDE textbook that derives the equations for the vibrating string in the manner described in section 2. Other notable exceptions include Zauderer's book [13], which derives the telegrapher's equation using correlated random walks, Lin and Segel's book [8], which derives the one-dimensional wave equation as a special case of elastic wave propagation, and Kevorkian's book [6], which derives the linear wave equation via an asymptotic expansion in the case of small amplitudes.

Derivations of the wave equation that make unnecessary assumptions are likely to give students the false impression that deriving PDEs is an ad hoc and unrealistic process. This impression causes students to be less comfortable deriving PDEs on their own and therefore less competent mathematical modelers. Given that the wave equation is the canonical example of a hyperbolic PDE, it is important that students

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be introduced to the wave equation in a flexible, physically realistic, and systematic way.

In this paper, we put forth the additional argument that this “honest” derivation of the wave equation provides students access to a variety of interesting applications. It also gives instructors an opportunity to demonstrate that PDEs are used by mathematicians and scientists in concert with other mathematical topics (multivariable calculus, ordinary differential equations, and asymptotic expansions, to name a few). In section 2, we present this derivation, followed by applications to strings, chains, and ropes.

2. A Derivation of the Wave Equation for Strings, Chains, and Ropes. Our goal in this section is to derive a general equation governing the motion of any one-dimensional elastic material, like a string, chain, or rope, situated in three dimensions. Smoothness of all functions is assumed where necessary.

Suppose that the center of a string, chain, or rope is a curve described parametrically by

$$\mathbf{x}(s,t) = \begin{bmatrix} x(s,t) \\ y(s,t) \\ z(s,t) \end{bmatrix},$$

such that continuously increasing s traces out the curve in $\mathbb{R}^3$. This curve may have finite or infinite length. Furthermore, let us imagine that each value of s refers to a specific point along the string, chain, or rope, in the same way that mile markers refer to specific locations along a highway. So, we can track how each part of the curve moves in time by fixing s and changing t. Since we have some freedom in this parametric dependence, for convenience, let us choose s to represent the arc-length along the curve in some reference or resting configuration. For example, a tautly held string of length L stretching from $(0,0,0)$ to $(L,0,0)$ could be described by

$$\mathbf{x}_{\text{rest}}(s) = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} \qquad \text{for } 0 \le s \le L.$$

When the parameter for a parametric curve measures arc-length, $\|\mathbf{x}_s\| \equiv \|\partial\mathbf{x}/\partial s\| = 1$. However, it is important to keep in mind that in nonequilibrium configurations of the curve, s may not correspond to arc-length as portions of the curve can be stretched ($\|\mathbf{x}_s\| > 1$) or compressed ($\|\mathbf{x}_s\| < 1$). The equations governing the motion of the curve describe how the curve will respond to elongation or compression and external forces.

Consider a short segment of the curve between s and $s + \Delta s$ shown in the free-body diagram (Figure 1). The radius and density of the string, chain, or rope can vary with s. Let $\rho(s)$ be the linear mass density of the material (mass per unit length), which can be thought of as the product of the cross-sectional area and the actual density (mass per unit volume) of the material. If Δs is small, the total mass of the short segment is

$$\int_s^{s+\Delta s} \rho(\zeta)d\zeta \approx \rho(s)\Delta s.$$

As we intend to invoke Newton's second law, we must enumerate all of the forces that act on this short segment of the curve. First, there are contact forces on the

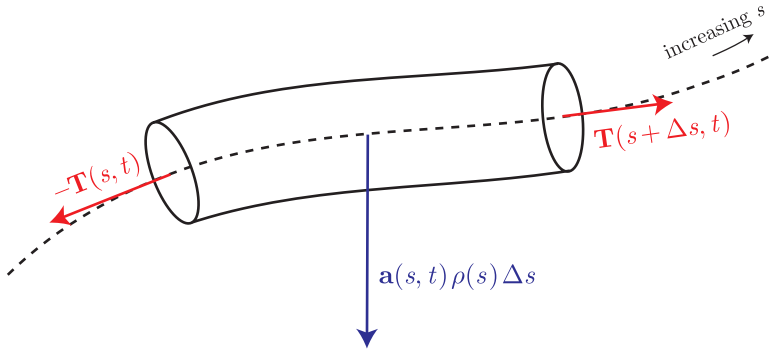

Fig. 1 *Free-body diagram of a small segment of the curve. The dashed curve is* $\mathbf{x}(s,t)$*, the centerline of the string, rope, or chain.*

segment due to the portion of the curve to the left and to the right. Let $\mathbf{T}(s,t)$ represent the force experienced by the segment to the left of s due to the segment right of s. At any moment, if the curve is cut at s, forces are required to keep both segments where they are. This thought experiment also implies that the force that the segment right of s experiences due to the segment left of s is $-\mathbf{T}(s,t)$. (A purely mathematical proof is given in [1].)

There may also be additional body forces acting on the segment, such as gravity or air resistance. Let the sum of all these body forces be $\mathbf{a}(s,t)\rho(s)\Delta s$. Then, using Newton's second law,

$$\mathbf{x}_{tt}(s,t)\rho(s)\Delta s = -\mathbf{T}(s,t) + \mathbf{T}(s+\Delta s,t) + \mathbf{a}(s,t)\rho(s)\Delta s.$$

After dividing both sides by Δs and allowing Δs to approach zero, we obtain the master wave equation

$$\rho(s)\mathbf{x}_{tt}(s,t) = \mathbf{T}_s(s,t) + \mathbf{a}(s,t)\rho(s). \tag{2.1}$$

Equation (2.1) alone is not enough to completely describe the motion of the curve, as it contains two unknown quantities, $\mathbf{x}$ and $\mathbf{T}$. More information about the material being described is needed. One quality that differentiates ropes, chains, and strings from rigid beams is that they do not resist bending. Consequently, the contact force $\mathbf{T}$ for such materials must always act in the direction of the tangent to the curve at s:

$$\mathbf{T}(s,t) = T(s,t)\frac{\mathbf{x}_s(s,t)}{\|\mathbf{x}_s(s,t)\|}. \tag{2.2}$$

The vector $\mathbf{x}_s/\|\mathbf{x}_s\|$ is a unit tangent vector, so the magnitude of $\mathbf{T}(s,t)$ is $T(s,t)$, which is also known as the *tension*.

In addition, a *constitutive relation* describing the response of the string, chain, or rope to compression or expansion is needed. Here are three examples of constitutive relations for elastic materials,[1] i.e., materials that return to their original shape after some applied deforming force is removed.

[1] *Elastic* materials have the property that tension is only a function of strain, $\|\mathbf{x}_s\|$, and position along the curve. Students may also enjoy learning about *viscoelastic* materials (like taffy), whose tension also depends on the time rate of change of strain.

The most common constitutive relation used in the derivation of the wave equation corresponds to a *perfectly elastic* material, in which tension is proportional to $\|\mathbf{x}_s\|$:

$$T(s,t) = E\|\mathbf{x}_s\|, \tag{2.3}$$

where the constant of proportionality, E, is known as Young's modulus. This constitutive relation is popular because it makes (2.1) a linear PDE. Rubber, stretched to several times its contracted length, satisfies this relation approximately [5]. However, notice that (2.3) predicts that the curve would have zero length in the absence of contact forces.

A more realistic assumption is that the material is *linearly elastic*, a material in which stress (force per unit area) is proportional to stretch, where

$$\text{stretch} = \frac{\text{change in length}}{\text{original length}} = \frac{\|\mathbf{x}_s\| - 1}{1}.$$

A corresponding constitutive relation would be

$$T(s,t) = k\left(\|\mathbf{x}_s\| - 1\right), \tag{2.4}$$

where k can be thought of as a spring constant.

A metal chain is an example of a curve that is *inextensible*, under reasonable forces. Inextensibilty is mathematically enforced by requiring that

$$\|\mathbf{x}_s\| = 1; \tag{2.5}$$

in other words, that s always corresponds to arc-length.

A brief history of this derivation of the governing equation for the vibrating string is given in [1]. This derivation is no more mathematically complicated than the common derivation of the one-dimensional wave equation, except that it encourages students to gain more familiarity with concepts from multivariable calculus such as vectors, parametric curves, and arc-length. Because the resulting equations are direct consequences of physical principles, this derivation builds students' physical intuition and facility for mathematical modeling. Finally, exposing students to a variety of constitutive relationships gives them opportunities for exploration.

3. Plucked Strings and Rubber Tubing. The canonical wave equation problem in introductory PDE courses is the plucked string problem: How does the profile of a tautly held guitar string evolve after it is plucked? Once (2.1) has been derived, this linear problem follows very naturally if we assume that the guitar string is perfectly elastic. (This is the most common assumption used in PDE textbooks, allowing for variations in terminology.)

For example, let us assume that the tautly held guitar string has uniform material properties and a resting length of L. In its resting configuration, the string is parameterized by

$$\mathbf{x}_{\text{rest}}(s) = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix} \qquad \text{for } 0 \le s \le L.$$

We'll assume that when the guitar string is plucked, the effect of gravity can be neglected. Substituting (2.3) into (2.1) and taking $\mathbf{a} = \mathbf{0}$, we get three linear PDEs,

$$x_{tt} = c^2 x_{ss},$$
$$y_{tt} = c^2 y_{ss},$$
$$z_{tt} = c^2 z_{ss},$$

where $c^2 = E/\rho$. Because these three equations are decoupled, the motions in each dimension are independent. If the string is plucked in such a way that its initial configuration lies in the x-z plane, it will always lie in this plane, since $y(s,t) = 0$ will be the solution to the PDE for y. This reduced problem can then be solved by prescribing the boundary conditions $x(0,t) = z(0,t) = z(L,t) = 0$ and $x(L,t) = L$ and initial data for $x(s,0)$, $x_t(s,0)$, $z(s,0)$, and $z_t(s,0)$. If the string is released from rest, then $x_t(s,0) = 0$ and $z_t(s,0) = 0$.

This problem provides instructors with an excellent opportunity to solve the basic wave equation with both homogeneous and inhomogeneous boundary conditions. However, if one does not wish to solve two separate PDE problems or deal with inhomogeneous boundary conditions, one can eliminate the x-PDE problem by assuming that each point on the string has no initial horizontal displacement and velocity. Specifically, if one assumes that $x(s,0) = s$ and $x_t(s,0) = 0$, then $x(s,t) = s$ for all time. This allows the z-PDE to be rewritten in the more familiar form

$$z_{tt} = c^2 z_{xx},$$

where z is the vertical displacement of the string.

Students can extend this problem by including air resistance or gravity, or by assuming a linearly elastic string. Another variant of the plucked string problem is the problem in which a piece of perfectly elastic rubber tubing is stretched to length L, with one end fixed and the other attached to a rotating wheel. One set of equations describing this scenario is

$$\mathbf{x}_{tt} = c^2 \mathbf{x}_{ss}$$

with

$$\mathbf{x}(0,t) = \begin{bmatrix} \alpha(1-\cos(\omega t)) \\ \alpha\sin(\omega t) \\ 0 \end{bmatrix}, \quad \mathbf{x}(L,t) = \begin{bmatrix} L \\ 0 \\ 0 \end{bmatrix}, \quad \mathbf{x}(s,0) = \begin{bmatrix} s \\ 0 \\ 0 \end{bmatrix}, \quad \text{and} \quad \mathbf{x}_t(s,0) = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}.$$

Here, α is the radius of the wheel and ω is the angular frequency of the wheel's rotation.

As this problem has inhomogeneous boundary conditions, solving it requires eigenfunction expansions and some facility with inhomogeneous ODEs. It also has the feature that certain values of ω lead to resonant excitation (see Figure 2). Movies of resonant and nonresonant behavior can be found at http://www.math.hmc.edu/~dyong/strings/.

4. Vibrational Modes of a Hanging Chain. The hanging chain problem is not only a good example of the power of mathematical modeling, but also an excellent way for students to become more familiar with Bessel functions besides studying problems involving circular geometries [12].

Imagine a heavy chain with linear mass density ρ and length L, hanging by its own weight from one of its ends. The motion of the chain is dominated by gravity, so we substitute $\mathbf{a}(s,t) = -g\mathbf{k}$ in (2.1) and use (2.5) to get

$$\rho\mathbf{x}_{tt}(s,t) = \frac{\partial}{\partial s}\left[T(s,t)\mathbf{x}_s\right] - \begin{bmatrix} 0 \\ 0 \\ \rho g \end{bmatrix}. \tag{4.1}$$

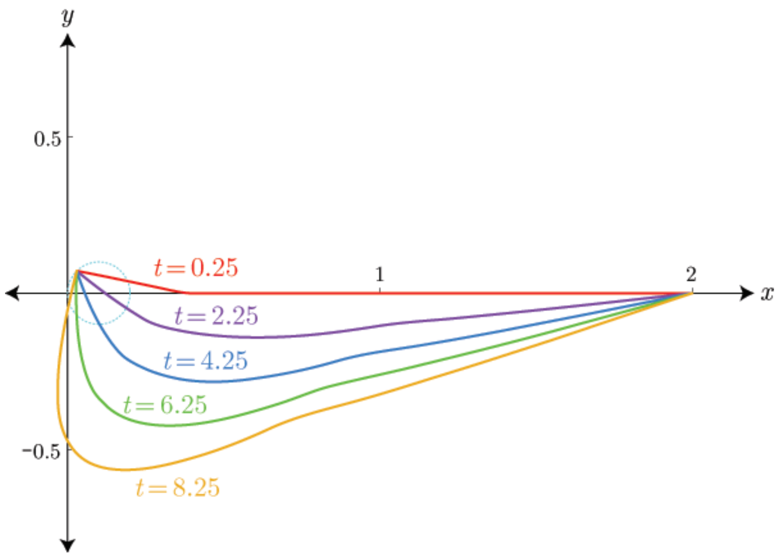

Fig. 2 *Snapshots of a piece of rubber tubing at different times, showing resonant behavior* ($\alpha = 0.1$, $\omega = \pi$, $L = c = 2$).

In its resting configuration, let the chain have the parameterization

$$\mathbf{x}_{\text{rest}}(s) = \begin{bmatrix} 0 \\ 0 \\ s \end{bmatrix} \qquad \text{for } 0 \le s \le L.$$

The parameter value $s = L$ corresponds to the end of the chain that is fixed, and $s = 0$ corresponds to the free end of the chain. Since the chain is fixed at $s = L$,

$$\mathbf{x}(L, t) = \begin{bmatrix} 0 \\ 0 \\ L \end{bmatrix}, \tag{4.2}$$

and because the chain must not have any tension at its free end,

$$\mathbf{T}(0, t) = T(0, t)\mathbf{x}_s(0, t) = \mathbf{0}. \tag{4.3}$$

Using this fact and (4.1), we find the resting (time-independent) tension of the chain to be $T_{\text{rest}}(s) = \rho g s$, which confirms our intuition that the resting tension of the chain is due to its own weight.

As the chain is inextensible, we will use (2.5), which can be more conveniently written as

$$\|\mathbf{x}_s\|^2 = \mathbf{x}_s \cdot \mathbf{x}_s = 1. \tag{4.4}$$

Since (4.4) is nonlinear, we will linearize[2] the governing equations around the resting state to obtain linear equations. Physically speaking, we are considering motions of

[2]Depending on the degree of sophistication of their students, instructors can use either formal asymptotic expansions or the simplified approach that we use here.

the chain that are very close to the resting, perfectly vertical state of the chain. To that end, let

(4.5a) $$\mathbf{x}(s,t) = \mathbf{x}_{\text{rest}}(s) + \epsilon\tilde{\mathbf{x}}(s,t) + \cdots$$

(4.5b) $$T(s,t) = T_{\text{rest}}(s) + \epsilon\tilde{T}(s,t) + \cdots ,$$

where ϵ is a positive number, much less than 1. The terms $\tilde{\mathbf{x}}$ and $\tilde{T}$ are the *perturbations* to the resting state, scaled by ϵ.

Substitute (4.5a) and (4.5b) into (4.3), (4.4), and (4.1), and rearrange terms according to their power of ϵ. Terms that are multiplied by ϵ^2 should be discarded as they are much smaller than terms multiplied by ϵ, and we have already discarded such terms in (4.5). All of the $O(1)$ terms (terms that are not multiplied by ϵ) should cancel each other out since the resting state of the chain is itself a solution of the governing equations.

Small movements of the chain away from the resting configuration are governed by the remaining $O(\epsilon)$ terms. The $O(\epsilon)$ terms from (4.3) imply that

$$\tilde{T}(0,t) = 0.$$

The $O(\epsilon)$ terms from (4.4) and (4.2) imply that

$$\tilde{z}(s,t) = 0,$$

meaning that to this level of approximation, points on the chain do not move vertically; the motion is largely in the x- and y-directions. When the components of the $O(\epsilon)$ terms in (4.1) are written out, we obtain the equations

(4.6a) $$\rho\tilde{x}_{tt} = \frac{\partial}{\partial s}\left(\rho g s\,\tilde{x}_s\right),$$

(4.6b) $$\rho\tilde{y}_{tt} = \frac{\partial}{\partial s}\left(\rho g s\,\tilde{y}_s\right).$$

(The z-component implies that $\tilde{T}(s,t) = 0$, since $\tilde{z} = 0$.) The motions of the chain in the x- and y-directions are decoupled at this level of approximation, so we can treat them in the same way.

Using separation of variables, the resulting linear boundary value problem,

$$\begin{aligned} \tilde{x}_{tt} &= \tfrac{\partial}{\partial s}\left(gs\,\tilde{x}_s\right) \\ &\text{subject to } \tilde{x}(L,t) = 0 \\ &\quad\text{and } |\tilde{x}(0,t)| < \infty, \end{aligned}$$

has product solutions[3]

$$\tilde{x}(s,t) = J_0\left(z_n\sqrt{\frac{s}{L}}\right)\left[\alpha\cos\left(\frac{z_n t}{2}\sqrt{\frac{g}{L}}\right) + \beta\sin\left(\frac{z_n t}{2}\sqrt{\frac{g}{L}}\right)\right],$$

where α and β are any constants and z_n is the nth positive root of $J_0(z)$. Each value of n corresponds to a different vibrational mode of the hanging chain, the first few of which are shown in Figure 3.

[3]Hint: After separating variables, the ODE involving s can be made to look like Bessel's differential equation using the change of variables $s = r^2$.

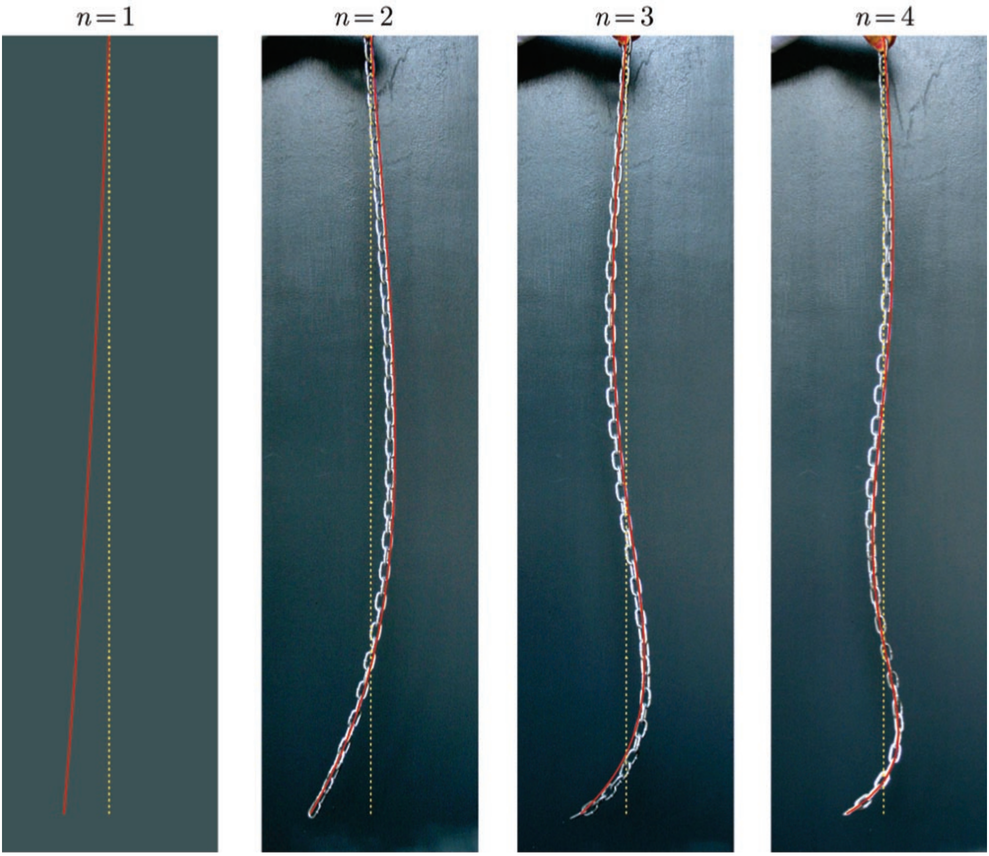

Fig. 3 *First four vibrational modes of the hanging chain; analytic solution shown in red. Photos courtesy of Kevin Mapp.*

As Lamb pointed out in [7], this solution correctly predicts not only the shapes of the modes, but also their frequencies. The period of the nth mode is

$$P_n = \frac{4\pi}{z_n}\sqrt{\frac{L}{g}}.$$

Verifying the analytically predicted frequencies of the vibrational modes of a hanging chain is an easy and fun classroom demonstration. Though the motions in the x- and y-directions are decoupled to this level of approximation, it is easier to generate these modes by spinning the top of a hanging chain, rather than moving it from side to side [12]. See http://www.math.hmc.edu/˜dyong/strings/ for movies taken during one of these demonstrations.

Students can explore this problem further by examining the initial value problem for the hanging chain problem (see [2] and [4]) or what happens at the $O(\epsilon^2)$ level of approximation. They might also be interested to read the wonderful paper on the dynamics of the driven hanging chain by Belmonte et al. [3].

5. Shape of a Jump Rope. The arched curves of a catenary, a suspension bridge, and a jump rope look similar, but they are subtly different because of differences

Fig. 4 *The shape of the main cables of the Golden Gate Bridge is very close to a parabola. Photo courtesy of Patricia A. Browne.*

in the forces acting on them. All three cases involve inextensible materials, and gravity is the dominant force responsible for the shape of a catenary (hyperbolic cosine) and a suspension bridge (parabola; see Figure 4). The difference between a catenary, the curve made by an inextensible chain suspended from two points, and a suspension bridge, is that the cables that support a suspension bridge are uniformly spaced horizontally, whereas gravity acts uniformly on each segment of a catenary. Mathematically speaking, the difference is whether the applied load is an integral involving dx or ds.

The shape of an inextensible jump rope in motion is different from the former two cases because at sufficiently high rotation rates, the dominant force is a centripetal force, which is stronger at points farther from the axis of rotation. As it turns out, the shape of a jump rope does not have a simple closed-form description like the catenary and suspension bridge, but we can determine its shape nonetheless.

Let ω be the angular velocity of the rotation and suppose that in that rotating frame of reference, the shape of the jump rope remains static in the x-y plane. Let us orient the fixed ends of the jump rope at $(0,0)$ and $(H,0)$, with the x-axis being the axis of rotation. In the rotating frame of reference, the apparent centrifugal "force" acting on a short segment of the rope spanned by Δs is $\omega^2 y(s)\rho(s)\Delta s\mathbf{j}$, so (2.1) becomes

$$\mathbf{0} = \frac{\partial}{\partial s}\left(T(s)\mathbf{x}_s\right) + \omega^2 y(s)\rho(s)\mathbf{j}, \tag{5.1}$$

which must be solved in conjunction with the inextensibility constraint (2.5). One way to solve this problem is to eliminate s and T and express y as a function of x. When the mass density is constant, the resulting differential equation can be solved using elliptic functions [9]; we do not discuss it further here.

An alternative way to attack this nonlinear ODE is to write it in terms of θ, the angle that the tangent to the curve makes with the x-axis. Let $x_s = \cos(\theta(s))$ and $y_s = \sin(\theta(s))$, so that the inextensibility constraint is automatically satisfied. In addition, the curvature of the jump rope becomes $|\theta'(s)|$. The two components of (5.1) become

$$0 = \frac{\partial}{\partial s}\left[T(s)\cos(\theta)\right], \tag{5.2a}$$

$$0 = \frac{\partial}{\partial s}\left[T(s)\sin(\theta)\right] + \omega^2\rho(s)\int_0^s \sin(\theta(\tau))d\tau. \tag{5.2b}$$

Equation (5.2a), when integrated, becomes

$$T(s) = \frac{C}{\cos(\theta)}, \tag{5.3}$$

where C is a constant. This equation tells us that the tension of the rope is minimized when $\theta = 0$, which occurs where the jump rope is farthest from its axis of rotation. That minimum tension is C. Using (5.3) in (5.2b), we get

$$0 = \frac{C\,\theta'(s)}{\rho(s)\cos^2(\theta(s))} + \omega^2 \int_0^s \sin(\theta(\tau))d\tau. \tag{5.4}$$

Substituting $s = 0$ into (5.4) gives $\theta'(0) = 0$, or that the jump rope has zero curvature at $s = 0$.

As (5.4) is a nonlinear equation, we will approximate its solutions numerically. To that end, we differentiate (5.4) once more to eliminate the integral. The system of equations to be solved is

$$x' = \cos\theta, \tag{5.5a}$$

$$y' = \sin\theta, \tag{5.5b}$$

$$\theta'' = -2(\theta')^2 \tan\theta + \frac{\rho'(s)}{\rho(s)}\theta' - \frac{\omega^2\rho(s)}{C}\sin\theta\cos^2\theta, \tag{5.5c}$$

subject to the boundary conditions $x(0) = 0$, $x(L) = H$, $y(0) = 0$, $y(L) = 0$, and $\theta'(0) = 0$. The minimum tension, C, and $\theta(0)$, the angle between the jump rope and the x-axis at $s = 0$, are both unknown and can be determined using a simple shooting method.

Different rotational modes of the jump rope are possible from this nonlinear boundary value problem, as shown in Figure 5. In fact, it is easy to see that the second mode (possessing one node in the middle of the jump rope) is made from two copies of the fundamental mode, scaled by a factor of 1/2. This formulation of the problem makes it easy to consider jump ropes with variable densities. An example of one such jump rope is shown in Figure 5.

One extension of this problem is to ask students why the shape of the jump rope becomes more sinusoidal as $\max y(s)/H$ tends to zero, as seen in Figure 5. This extension involves linearizing the ODEs and leads to a nice discussion about the differences between linear and nonlinear boundary value problems.

Another extension of this problem is to allow the jump rope to be linearly elastic; in other words, to consider the rotating telephone cord problem. Intuitively, we know that increasing the angular velocity of rotation increases the amplitude of the telephone cord's motion. This is different from the inextensible case, in which the angular velocity affects the tension of the jump rope rather than its shape.

A more difficult extension is to consider the temporal dynamics of the jump rope. For example, one can introduce a periodic forcing to simulate the effect of the jump rope hitting the ground periodically.

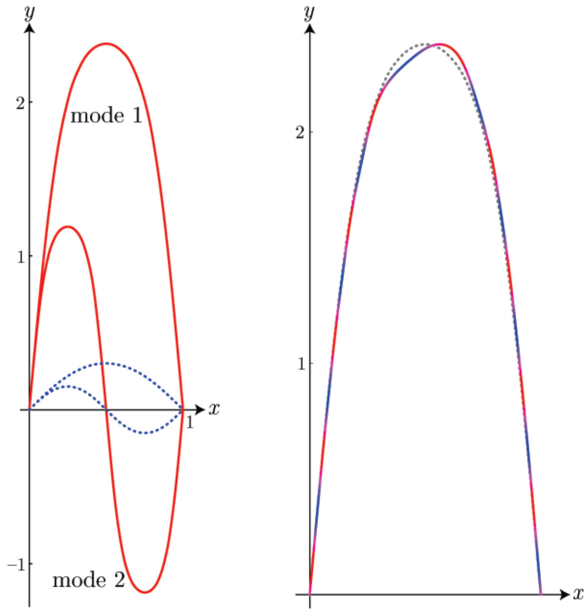

Fig. 5 *On the left, first two modes of a jump rope with $\rho(s) = 1$ and $\omega = 1$, for $L = 5$ (solid) and $L = 1.2$ (dotted). On the right, jump rope with $L = 5$ and $\omega = 1$, constant mass density $\rho(s) = 1$ (dotted) and variable mass density $\rho(s) = 5 + 4\sin(4\pi s)$ (red corresponds to $\rho = 9$, blue $\rho = 1$).*

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