1. If $\varphi$ and $\psi$ are 1-forms in $\mathbb{R}^3$, the wedge product $\varphi \wedge \psi$ is a 2-form on $\mathbb{R}^3$ such that

$$(\varphi \wedge \psi)(v, w) = \varphi(v)\psi(w) - \varphi(w)\psi(v)$$

for all pairs $v, w$ of tangent vectors in $\mathbb{R}^3$. Prove that $\varphi \wedge \psi$ is a skew-symmetric 2-form.

2. Let $p$ be a point of a regular surface $S$, and let $x : U \subset \mathbb{R}^2 \to S$, $y : V \subset \mathbb{R}^2 \to S$ be two parameterizations of $S$ such that $p \in x(U) \cap y(V) = W$. Show that the “change of parameters” $h = x^{-1} \circ y : y^{-1}(W) \to x^{-1}(W)$ is a diffeomorphism; that is, $h$ is differentiable and has a differentiable inverse $h^{-1}$. In other words, if $x$ and $y$ are given by

$$x(u, v) = (x(u, v), y(u, v), z(u, v)),$$

$$y(\xi, \eta) = (x(\xi, \eta), y(\xi, \eta), z(\xi, \eta)),$$

then the change of coordinates $h$, given by

$$u = u(\xi, \eta) \quad v = v(\xi, \eta), \quad (\xi, \eta) \in y^{-1}(W),$$

has the property that the functions $u$ and $v$ have continuous partial derivatives of all orders, and the map $h$ can be inverted, yielding

$$\xi = \xi(u, v) \quad \eta = \eta(u, v), \quad (u, v) \in x^{-1}(W),$$

where the functions $\xi$ and $\eta$ also have partial derivatives of all orders. Since

$$\frac{\partial(u, v)}{\partial(\xi, \eta)} \cdot \frac{\partial(\xi, \eta)}{\partial(u, v)} = 1,$$

this implies that the Jacobian determinants of both $h$ and $h^{-1}$ are continuous everywhere.

3. Show that, in the basis

$$e_i' = c_i^j e_j$$

the coefficients $b_i'$ of a functional $B \in T_1^1(V)$ are expressed by the formula

$$b_i' = c_i^j b_j'.$$

4. Let

$$B = \begin{pmatrix} b_1^1 & \cdots & b_n^1 \\ \vdots & \ddots & \vdots \\ b_1^n & \cdots & b_n^n \end{pmatrix}, \quad x = \begin{pmatrix} x^1 \\ \vdots \\ x^n \end{pmatrix}, \quad \xi = (\xi_1, \ldots, \xi_n).$$

Show that $B' = C^{-1}BC$.

5. Express the following vector fields (i) in terms of the cylindrical frame field (with coefficients in terms of $r$, $\theta$, and $z$) and (ii) in terms of the spherical frame field (with coefficients in terms of $\rho$, $\theta$, $\varphi$).

(a) $U_1$
(b) Let \( O \) be all of \( E^3 \) except the \( z \) axis and the circle \( C \) of radius \( R \) in the \( xy \) plane. The toroidal coordinate functions \( \rho, \theta, \varphi \) are defined on \( O \) as suggested in figure 1, so that

\[
\begin{align*}
x &= (R + \rho \cos \varphi) \cos \theta \\
y &= (R + \rho \cos \varphi) \sin \theta \\
z &= \rho \sin \varphi.
\end{align*}
\]

If \( E_1, E_2, \) and \( E_3 \) are unit vector fields in the direction of increasing \( \rho, \theta, \) and \( \varphi, \) respectively, express \( E_1, E_2, \) and \( E_3 \) in terms of \( U_1, U_2, \) and \( U_3, \) and prove that it is a frame field.

(c) Classical vector analysis avoids the use of differential forms on \( \mathbb{R}^3 \) by converting 1-forms and 2-forms into vector fields by means of the following one-to-one correspondences.

\[
\begin{align*}
f_1 dx^1 + f_2 dx^2 + f_3 dx^3 &\leftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3 \\
f_1 dx^2 \wedge dx^3 + f_2 dx^3 \wedge dx^1 + f_3 dx^1 \wedge dx^2 &\leftrightarrow f_1 \varepsilon_1 + f_2 \varepsilon_2 + f_3 \varepsilon_3
\end{align*}
\]

Vector analysis uses three basic operations based on partial differentiation:

1. **Gradient** of a function \( f \):

\[
\text{grad}(f) = \sum_{i=1}^{3} \frac{\partial f}{\partial x^i} \varepsilon_i
\]

2. **Curl** of a vector field \( v = \sum_{i=1}^{3} v^i(x) \varepsilon_i \):

\[
\text{curl}(v) = \left( \frac{\partial v^3}{\partial x^2} - \frac{\partial v^2}{\partial x^3} \right) \varepsilon_1 + \left( \frac{\partial v^1}{\partial x^3} - \frac{\partial v^3}{\partial x^1} \right) \varepsilon_2 + \left( \frac{\partial v^2}{\partial x^1} - \frac{\partial v^1}{\partial x^2} \right) \varepsilon_3
\]

3. **Divergence** of a vector field \( v = \sum_{i=1}^{3} v^i(x) \varepsilon_i \):

\[
\text{div}(v) = \sum_{i=1}^{3} \frac{\partial v^i}{\partial x^i}
\]

Prove that all three operations may be expressed in terms of exterior derivatives as follows:

1. \( df \leftrightarrow \text{grad}(f) \)
2. If \( \varphi \) is a 1-form and \( \varphi \leftrightarrow v, \) then \( d\varphi \leftrightarrow \text{curl}(v). \)
3. If \( \eta \) is a 2-form and \( \eta \leftrightarrow v, \) then \( d\eta \leftrightarrow \text{div}(v) dx^1 \wedge dx^2 \wedge dx^3. \)

Show that the identities

\[
\text{curl(\text{grad}(f))} = 0
\]
\[
\text{div(\text{curl}(v))} = 0
\]

follow from the fact that \( d^2 = 0. \)

(d) Let \( f \) and \( g \) be real-valued functions on \( \mathbb{R}^2. \) Prove that

\[
df \wedge dg = \begin{vmatrix} \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} \\ \frac{\partial g}{\partial x} & \frac{\partial g}{\partial y} \end{vmatrix} dx \wedge dy.
\]