Do each of the following problems.

(1) Study thoroughly the handout of “Topics on Proving Methods and Techniques”. Please carry out details especially for those problems you feel that you need to carry them out carefully. You may turn in the handout with your fill-in as the answers to this assignment.

(2) Let $X$ be a linear space over field $K$. Let $0$ is the zero element of $K$ and $0$ is the zero element of $X$. Using only the properties given on Page 1 of Lax (and, of course, basic properties of a field), carefully prove each of the following.
   (a): $0x = 0$ for every $x \in X$.
   (b): $k0 = 0$ for every $k \in K$.
   (b): For every $k \in K$ and $x \in X$, $kx = 0 \Rightarrow k = 0$ or $x = 0$.

(3) Prove the following Lemma.

**Lemma 0.1.** (Replacement Lemma). Let $X$ be a linear space over field $K$, and let $S$ be a linearly independent subset of $X$. Let $x_0 \in \text{span}(S)$ with $x_0 \neq 0$. Prove that there exists $x_1 \in S$ such that the set $S' = (S\setminus \{x_1\}) \cup \{x_0\}$ is a basis for $\text{span}(S)$.

(4) Let $X$ be a linear space over field $K$, and suppose that $B$ is a basis for $X$ with $B$ containing $n$ elements. Let $B'$ be any basis for $X$. Use to Replacement Lemma to carefully prove that $B'$ is a finite set containing exactly $n$ elements.