Do each of the following problems.

(1) Study the handout on “A Review for Elementary Linear Algebra”. Study all the problems on the Sample Midterm Exams I and II, and the Sample Final Exam by using the answer keys. Select two problems which you think you would like to carry out in details, and turn them in.

(2) For following problem, you might find Sample Midterm I, (3A) to be useful.

(a): Let \( \text{Symm}M_n(K) \) and \( \text{Skew}M_n(K) \) denote the set of symmetric matrices and the set of skew symmetric matrices respectively. Show that \( \dim(M_n) = \dim(\text{Symm}M_n(K)) + \dim(\text{Skew}M_n(K)) \).

(b): Show that every \( n \times n \) matrix can be uniquely decomposed as a sum of a symmetric matrix and a skew symmetric matrix.

(c): Conclude that \( M_n = \text{Symm}M_n(K) \oplus \text{Skew}M_n(K) \).

(3) Exercise 12, Chapter 1, Page 5 of Lax.

(4) Exercise 15, Chapter 1, Page 6 of Lax.

(5) Exercise 16, Chapter 1, Page 7 of Lax.

(6) Let \( X \) be a vector space over a field \( K \). Let \( Y \) and \( Z \) be subspaces of \( X \). Is the following statement true or false? If it is true, prove it. Otherwise, give at least one counter example.

(a): \( X = Y + Z \) is a direct sum if and only if \( Y \cap Z = \{0\} \).

(b): \( X \) is a direct sum of \( Y \) and \( Z \) if and only if \( \dim X = \dim Y + \dim Z \).

(7) Let \( S \) be a set of unknown number of vectors in a finite dimensional vector space. Show that \( S \) is a basis of \( V \) if every vector of \( V \) can be written in one and only one way as a linear combination of the vectors in \( S \). (Please review the handout on when does a subset of vector space \( V \) form a basis of \( V \) ?)

(8) Prove the lemma we used in class: Let \( V \) be a finite-dimensional vector space over the field \( K \) and let \( \{v_1, v_2, ..., v_n\} \) be an ordered basis for \( V \). Let \( W \) be a vector space over the same field \( K \) but may be with a different dimension and let \( \{w_1, w_2, ..., w_n\} \) be any vectors in \( W \). Then there is precisely one linear transformation \( T \) from \( V \) into \( W \) such that

\[
Tv_i = w_i, i = 1, ..., n.
\]

(9) (a): Let \( l \in (K^n)' = \) the dual of the vector space of \( n \)-tuples from \( K \). Prove that there exist unique \( a_1, a_2, ..., a_n \in K \) such that

\[
l(x_1, x_2, ..., x_n) = \sum_{i=1}^{n} a_i x_i,
\]

(b): Show that (a) can be generalized to any finite dimensional vector space \( V \) if we fix a basis \( \{v_1, v_2, ..., v_n\} \). That is to show that every linear functional on \( V/K \) is of the form

\[
l(v) = \sum_{i=1}^{n} a_i x_i,
\]
where

\[ v = \sum_{i=1}^{n} x_i v_i \]

Extra credit problems (Optional).

(1) In problem 6, if there is any statement which is not correct, could you add in some condition(s) so that the result statement is true? Explain your answer.

(2) Let \( X \) be a vector space over a field \( K \). Let \( Y \) and \( Z \) be subspaces of \( X \). Suppose \( W \) is a subspace of \( X \) containing both \( Y \) and \( Z \). Show that \( W \) must contain \( Y + Z \). In this sense, show that \( Y + Z \) is the minimal subspace containing both \( Y \) and \( Z \).