Do each of the following problems.

(1) Exercise 2, Chapter 5, Page 35 of Lax.

(2) Exercise 3, Chapter 5, Page 35 of Lax. Hint: You can easily get the result from properties in Lax prior to Exercise 3. Or, can use the following approach. Let $p_0$ denote the identity permutation. Suppose that

$$p_k = t_k \circ t_{k-1} \circ \cdots \circ t_1$$

where $t_i$ is a transposition, $k$ is odd and $k$ is minimal with $(*)$ holding. Using the notation introduced below, write $t_k = (a b)$. Show that $t_k \circ t_{k-1}$ can be written in the form $(y z)(a x)$ where $a \notin \{x, y, z\}$. Use this to get a contradiction on $(*)$.

(3) Exercise 4, Chapter 5, Page 37 of Lax. (Note: This shows that if we start with (16) as the definition of $D$, then all the remaining results on determinants hold. Consequently, we could get the definition of $D$ and hence of the determinant without use of geometric arguments.)

(4) Exercise 5, Chapter 5, Page 37 of Lax.

(5) Exercise 6, Chapter 5, Page 39 of Lax.

(6) Exercise 7, Chapter 5, Page 41 of Lax.

(7) Exercise 8, Chapter 5, Page 42 of Lax.

(8) Let $A$ be an $n \times n$ matrix and $A_{ij}$ be the $i,j$ th minor of $A$ (defined on Page 39 of Lax.). Let $c_{ij} = (-1)^{i+j} \det(A_{ij})$ for $1 \leq i, j \leq n$, and define $C$ to be the $n \times n$ matrix $(c_{ij})$; i.e. the $i,j$ th entry of $C$ is the element $c_{ij}$. Prove that

$$CTA = \det(A)I$$

where $I$ is the $n \times n$ identity matrix.

**Notation**: Here are some usual notations for permutation: We will write $(p_1 \ p_2 \ \ldots \ p_k)$, where $p_1, p_2, \ldots, p_n$ are distinct elements of the set $\{1, 2, 3, \ldots, n\}$, to represent the permutation $p$ satisfying that

$$p_2 = p(p_1), p_3 = p(p_2), \ldots, p_n = p(p_{n-1}), p_1 = p(p_n)$$

and $p(i) = i$ for $i \notin \{p_1, p_2, p_3, \ldots, p_n\}$.

For example, if $n = 5$, then $(1 \ 3 \ 4)$ would represent the permutation which in Lax’s notation would be

\[
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 \\
3 & 2 & 4 & 1 & 5
\end{pmatrix}
\]

A permutation of this form is called a cycle. As is illustrated in the following examples, any permutation can be written as a composition of disjoint cycles. The permutations of Lax’s Example 1, Page 34, can be written as

$$p = (1 \ 2 \ 4 \ 3), \ p^2 = (1 \ 4)(2 \ 3), \ p^3 = (1 \ 3 \ 4 \ 2), \ p^4 = (1)(2)(3)(4) = (1) = (2) = (3) = (4)$$

Working left to right, you might verify the following result

$$\begin{pmatrix}
1 & 2 & 3 \\
2 & 1 & (4 \ 5 \ 3)(3 \ 4)
\end{pmatrix} \circ \begin{pmatrix}
2 & 1 \\
1 & (4 \ 5 \ 3)(3 \ 4)
\end{pmatrix} \circ \begin{pmatrix}
4 & 5 & 3 \\
4 & 5 & 3
\end{pmatrix} \circ \begin{pmatrix}
3 & 4 \\
3 & 4
\end{pmatrix} = \begin{pmatrix}
3 & 5 & 1 \\
3 & 5 & 1
\end{pmatrix}$$

Normally, instead of writing $(1 \ 2 \ 3)(2 \ 1)(4 \ 5 \ 3)(3 \ 4)$, we simply write $(1 \ 2 \ 3)(2 \ 1)(4 \ 5 \ 3)(3 \ 4)$. 