Problem Set VI - Due: Tuesday 11:00 p.m., March 1.

Do each of the following problems

(1) Let $A$ be an invertible $n \times n$ matrix. Prove that $A^{-1}$ can be written as a polynomial in $A$ of degree at most $n - 1$. That is, prove that there are scalars $a_i$ such that

$$A^{-1} = a_{n-1}A^{n-1} + a_{n-2}A^{n-2} + \cdots + a_1A + a_0I$$

(2) Consider the following matrices:

$$A = \begin{pmatrix} -2 & 0 \\ 0 & -2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -2 \\ -2 & 0 \end{pmatrix}, \quad C = \begin{pmatrix} 3 & 1 \\ -1 & 1 \end{pmatrix}$$

(a): Determine the characteristic and minimal polynomials of each of these matrices.

(b): Determine the eigenvectors and generalized eigenvectors for each of the matrices $A, B, C$.

(3) Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

A straightforward calculation shows that the characteristic polynomials

$$p_A(s) = p_B(s) = p_C(s) = (x - 2)^2(x - 3).$$

(You need not verify that!)

(a): Determine the minimal polynomials $m_A(s), m_B(s),$ and $m_C(s)$.

(b): Determine the eigenvectors and generalized eigenvectors for each of the matrices $A, B, C$.

(4) Exercise 6, Chapter 6, Page 57 of Lax.

(5) Consider the following matrices:

$$A = \begin{pmatrix} 2 & -2 & 14 \\ 0 & 3 & -7 \\ 0 & 0 & 2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & -4 & 85 \\ 1 & 4 & -30 \\ 0 & 0 & 3 \end{pmatrix}, \quad C = \begin{pmatrix} 2 & 2 & 1 \\ 0 & 2 & -1 \\ 0 & 0 & 3 \end{pmatrix}$$

A straightforward calculation shows that the characteristic polynomials

$$p_A(s) = p_B(s) = p_C(s) = (x - 2)^2(x - 3).$$

(You need not verify that! Use Lax’s Theorem 12, page 57, to determine which of the matrices $A, B, C$ are similar.

(6) Consider the matrix

$$A = \begin{pmatrix} -1 & 0 & 1 & 0 \\ 2 & 1 & 2 & 1 \\ 0 & 0 & -1 & 0 \\ 4 & 0 & -6 & 1 \end{pmatrix}$$

Determine the Jordan Canonical Form for the matrix $A$. 

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(7) Consider the matrix

\[ B = \begin{pmatrix}
1 & 0 & 0 & 1 \\
2 & 1 & 0 & -4 \\
1 & 0 & 1 & -2 \\
0 & 0 & 0 & 1
\end{pmatrix} \]

Determine the Jordan Canonical Form for the matrix \( B \)

(8) Exercise 7, Chapter 6, Page 59 of Lax.

(9) Exercise 9, Chapter 6, Page 60 of Lax.