Do each of the problems 1, 2, 6, 7, 8 and choose two problems out of 3, 4, 5. The left one will be the extra credit problem.

(1) Exercise 1, Chapter 7, Page 64 of Lax. Note: What Lax really meant to say there is:

\[ \| x \| = \max \{ (x, y) : y \in K^n \text{ with } \|y\| = 1 \} \]

(2) Define \( \ell^2 = \left\{ \{z_k\}_{k=1}^{\infty} : z_k \in \mathbb{C} \text{ and } \sum_{k=1}^{\infty} |z_k|^2 < \infty \right\} \)

where for \( z = x + iy \), \( |z| = |x + iy| = \sqrt{x^2 + y^2} = \sqrt{z \cdot \overline{z}} \) where \( \overline{z} = x - iy \).

Define \((\cdot, \cdot) : \ell^2 \times \ell^2 \to \mathbb{C}\) by

\[ (* : \{z_k\}_{k=1}^{\infty}, \{w_k\}_{k=1}^{\infty}) = \sum_{k=1}^{\infty} z_k \overline{w_k} \]

(a): Prove that \( \ell^2 \) is a linear space over \( \mathbb{C} \); that is, prove that \( \ell^2 \) is a subspace of \( \mathbb{C}^{\infty} \) as defined earlier this semester. (Note: You may assume the standard properties and results about infinite series.)

(b): Prove that the definition given in (*) gives \( \ell^2 \) as an inner product space.

(3) Exercise 2, Chapter 7, Page 66 of Lax. Note: You may assume that both \( \mathbb{C} \) and \( \mathbb{R} \) are complete and locally compact (i.e., any Cauchy sequence in \( \mathbb{C} \) or \( \mathbb{R} \) will be convergent and any bounded sequence in \( \mathbb{C} \) or \( \mathbb{R} \) will have a convergent subsequence.)

(4) Prove that \( \ell^2 \) is not locally compact. That is, show that there exists a bounded sequence in \( \ell^2 \) that has no convergent subsequence.

(5) Exercise 3, Chapter 7, Page 70 of Lax.

(6) Exercise 4, Chapter 7, Page 70 of Lax.

(7) Let \( X \) be an inner product space over \( \mathbb{C} \). Prove that for \( x, y \in X \)

\[ |(x, y)| \leq \|x\| \cdot \|y\| \]

with equality \( \iff \) one of \( x \) or \( y \) is a scalar multiple of the other.

[Remark: As we discovered in class, we cannot directly use the idea of Lax’s proof of Schwarz’s Inequality, page 64. However, a minor modification in the proof will work. As a hint, let me suggest that you find a formula for \( c \in \mathbb{C} \) where \( x = cy \) with \( y \neq 0 \). If you follow this approach you will want to look at a quadratic polynomial where the variable of the polynomial only takes on real values.]

(8) Let \( X \) be an inner product space over \( \mathbb{C} \). Prove that for \( x, y \in X \)

\[ \|x + y\| \leq \|x\| + \|y\| \]

with equality \( \iff \) one of \( x \) or \( y \) is a non-negative real scalar multiple of the other.