Section 5.5  5, 7, 24, 29
Section 6.2  4, 13, 15, 17, 20

Sec 5.5  
5) If $T(u, v, w) = (3u-v, u-v+2w, 5u+3v-w)$ describe how $T$ transforms the unit cube $W = [0, 1] \times [0, 1] \times [0, 1]$.

We can write $T$ by using matrix multiplication:

$$T(u, v, w) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} u \\ v \\ w \end{bmatrix}$$

Note that if

$$A = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix}$$

Then $\det A = -26 \neq 0$.

So $T$ is one-to-one and onto, and $T$ maps parallelepipeds to parallelepipeds. In particular, the unit cube $W = [0, 1] \times [0, 1] \times [0, 1]$ is mapped onto some parallelepiped $W' = T(W^2)$ with volume $|\det A| = 26$, so $V = 26$.

The image of the vertices of the cube are:

$T(0, 0, 0) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} = (0, 0, 0)$

$T(0, 0, 1) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = (0, 2, -1)$

$T(1, 0, 0) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix} = (2, 0, 8)$

$T(0, 1, 0) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} = (-1, -1, 3)$

$T(1, 1, 0) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} = (3, 3, 4)$

$T(1, 0, 1) = \begin{bmatrix} 3 & -1 & 0 \\ 1 & -1 & 2 \\ 5 & 3 & -1 \end{bmatrix} \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix} = (3, 1, 5)$
To summarize,

$T(0,0,0) \rightarrow (0,0,0) \rightarrow (1,0,0) \rightarrow (2,1,5)$

$T(0,1,0) \rightarrow (-1,-1,3) \rightarrow (0,0,1) \rightarrow (0,2,-1)$

$T(1,1,0) \rightarrow (2,0,8) \rightarrow (1,0,1) \rightarrow (3,3,3)$

$T(0,1,1) \rightarrow (-1,1,2) \rightarrow (1,1,1) \rightarrow (2,2,2)$

with these vertices uniquely define the parallelepiped $D$.

7) Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be the transformation given by

$T(p,q,r) = (p \cos \theta \cos \phi, p \sin \theta, p \sin \phi)$

1) Determine $D = T(0^3)$, where $D = [0,1] \times [0, \pi] \times [0, 2\pi]$.

$D(x,y,z) = (p \cos \phi \sin \theta, p \sin \phi \sin \theta, p \cos \theta)$

Note that they correspond to spherical coordinates.

$p : [0, 1]$, $\theta : [0, \pi]$, $\phi : [0, 2\pi]$

$0 \leq p \leq 1$, $0 \leq \theta \leq \pi$, $0 \leq \phi \leq 2\pi$

in which we ranges for a uniform sphere with a radius of 1.

$D$ can thus be described as

$D = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1\}$
b) Define \( D \cap T (0^+) \) where
\[
D^+ = [0,1] \times [0, \pi/2] \times [0, \pi/2]
\]
\[
(x,y,z) = (r \sin \psi \cos \theta, r \sin \psi \sin \theta, r \cos \theta)
\]
\[
p \in [0,1] \quad \psi \in [0, \pi/2] \quad \theta \in [0, \pi/2]
\]
\[0 \leq p \leq 1 \quad 0 \leq \psi \leq \pi/2 \quad 0 \leq \theta \leq \pi/2
\]

Notice spherical coordinates, except now the ranges \( \psi \) and \( \theta \) are such that only the 1st octant portion of the unit sphere remains. (Half the \( \psi \), quarter the \( \theta \)). The range \( D^+ \) is then a quarter sphere in ending.

\( D \) can thus be described as
\[
D^+ = \{(x,y,z) \mid x^2 + y^2 + z^2 \leq 1, x, y, z \geq 0\}
\]
Determine \( D = T(D^\times) \), where \( D^\times = [\frac{1}{2}, 1] \times [0, \frac{\pi}{2}] \times [0, \frac{\pi}{2}] \).

\[
(x, y, z) = (\rho \sin \theta \cos \phi, \rho \sin \theta \sin \phi, \rho \cos \theta) \\
\rho: [\frac{1}{2}, 1] \\
\theta: [0, \frac{\pi}{2}] \\
\phi: [0, \frac{\pi}{2}] \\
\frac{1}{2} \leq \rho \leq 1 \\
0 \leq \theta \leq \frac{\pi}{2} \\
0 \leq \phi \leq \frac{\pi}{2}
\]

Note that because the polar images are the same as that of point \( b \), the image \( D \) will look like a quarter sphere.

However, the ranges of these spheres, determined by the image of \( \rho \),
only span from \( \frac{1}{2} \) to 1 instead of 0 to 1, making the sphere
hollow from \( 0 \) to \( \frac{1}{2} \).

Thus, the image of \( D \) is a quarter sphere of radius 1
with a hole of a size of a quarter sphere of radius \( \frac{1}{2} \).

\[ D \]

\[ T \]

\[ \rho \]

\[ z \]

\[ y \]

\[ x \]

\[ P \]

\[ D^\times \]

\[ \sim \]

\[ \{ (x, y, z) \mid \frac{1}{2} \leq x^2 + y^2 + z^2 \leq 1, x, y, z \geq 0 \} \]

The constant \( z \) can be described as
\[ z = \frac{1}{2} \to 1 \]

\[ z \text{ constant} \]
25) Determine the volume of the given integral, where W is the region bounded by the two planes \( x^2 + y^2 + z^2 = a^2 \) and \( x^2 + y^2 + z^2 = b^2 \), for \( 0 < a < b \).

Use spherical coordinates.

Let

\[
\begin{align*}
x &= \rho \sin \phi \cos \theta \\
y &= \rho \sin \phi \sin \theta \\
z &= \rho \cos \phi
\end{align*}
\]

Then

\[
\begin{align*}
x^2 + y^2 + z^2 &= \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \\
&= \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \\
&= \rho^2 \sin^2 \phi (\cos^2 \theta + \sin^2 \theta) + \rho^2 \\
&= \rho^2 \sin^2 \phi \cdot 1 + \rho^2 \\
&= \rho^2 \\
\sqrt{x^2 + y^2 + z^2} &= \rho \\
\end{align*}
\]

Hence, \( dV \)

\[
dV = d\rho \sin \phi \, d\phi \, d\theta
\]

The Jacobian

\[
\frac{\partial(x, y, z)}{\partial(\rho, \phi, \theta)} = \begin{vmatrix}
x & x \\
y & y \\
z & z
\end{vmatrix} = \begin{vmatrix}
\sin \phi \cos \theta & \rho \sin \phi \cos \theta & \rho \sin \phi \cos \theta
\sin \phi \sin \theta & \rho \sin \phi \sin \theta & \rho \sin \phi \sin \theta
\cos \phi & 0 & 0
\end{vmatrix}
\]

Using u-substitution, note the last row, the determinant is equal to

\[
\begin{align*}
\cos \phi \left( \rho^2 \cos^2 \phi \sin^2 \phi + \rho^2 \sin^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \cos^2 \theta \right) + \rho^2 \sin \phi \left( \rho^2 \cos^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \sin^2 \theta \right)
\end{align*}
\]

\[
= \rho^2 \cos \phi \left( \rho^2 \cos^2 \phi \sin^2 \phi \right) + \rho^2 \sin \phi \left( \rho^2 \cos^2 \phi \cos^2 \theta + \rho^2 \sin^2 \phi \sin^2 \theta + \rho^2 \cos^2 \phi \sin^2 \theta \right)
\]

\[
= \rho^2 \sin \phi \left( \cos^2 \phi \cos^2 \theta + \sin^2 \phi \sin^2 \theta \right)
\]

\[
= \rho^2 \sin \phi \left( \cos^2 \phi + \sin^2 \phi \right)
\]

\[
= \rho^2 \sin \phi
\]
We need to find \[ \frac{\partial(x, y, z)}{\partial(p, \theta, \phi)} \] and we can see that under the restriction of the spherical coordinates of $0 \leq \theta \leq \pi$, $\sin\phi$ will always be non-negative. Thus, the Jacobian will also be non-negative.

Therefore, the volume element in spherical coordinates is

\[ dV = \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta. \]

And by the change of variables formula,

\[
\iiint_{\mathbb{R}^3} f(x, y, z) \, dx \, dy \, dz = \iiint_{\mathbb{R}^3} f(f^{-1}(x, y, z)) \cdot \left| \det \frac{\partial (x, y, z)}{\partial (u, v, w)} \right| \, du \, dv \, dw
\]

Finally, \[ W = \int_{\text{Region bounded by two spheres}} \rho^2 \sin\theta \, d\rho \, d\phi \, d\theta. \]

The region is bounded by two spheres, one with radius $a$ and the other with radius $b$.

\[ a \leq \rho \leq b, \quad 0 \leq \phi \leq \pi, \quad 0 \leq \theta \leq 2\pi. \]
\[ \iiint_W (2 + x^2 + y^2) \, dV \]

where \( W \) is the region inside the sphere \( x^2 + y^2 + z^2 = 25 \) and above the plane \( z = 3 \).

Using cylindrical coordinates, the range of \( z \) is \( z = 3 \rightarrow z = \sqrt{25 - r^2} \).

Shading in the \( x-y \) plane:

\[ x^2 + y^2 = 25 \quad z = 3 \quad x^2 + y^2 + z^2 = 25 \]

where
\[
\begin{align*}
  x &= r \cos \theta \\
  y &= r \sin \theta \\
  z &= z
\end{align*}
\]

Determine \( dV \):

\[ dV = r \, dr \, d\theta \, dz \]

Determine Jacobian:

\[
\frac{\partial (x, y, z)}{\partial (r, \theta, z)} = \begin{vmatrix}
  r \cos \theta & r \sin \theta & 0 \\
  0 & r \cos \theta & 0 \\
  0 & 0 & 1
\end{vmatrix} = r \cos^2 \theta + r \sin^2 \theta = r.
\]
so we see that the volume element in cylindrical coordinates is
\[ dV = r \, dr \, d\theta \, dz \]

Recall that the cylindrical coordinate \( r \) is usually taken to be non-negative by convention. In our case, \( r = 0 \rightarrow 5 \), which is non-negative.

\[ \frac{\partial (x, y, z)}{\partial (r, \theta, z)} = r. \]

By the change of variables formula,

\[ \iiint_W (2 + x^2 + y^2) \, dV = \iiint_W (2 + r^2) \, r \, dr \, d\theta \, dz \]

\[ = \int_0^4 \int_0^5 \int_3^{r+3} (2 + r^2) \, 2r + r^2 \, dz \, dr \, d\theta \]

\[ \Rightarrow \frac{656 \pi}{5} \]

\[ \Rightarrow \frac{656 \pi}{5} \]
Sec 6.2.

4) Verify Green's theorem for the given vector field

\[ \mathbf{F} = M(x,y) \mathbf{i} + N(x,y) \mathbf{j} \]

over region \( D \) by calculating both

\[ \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA \]

and

\[ \oint_C \mathbf{F} \cdot d\mathbf{r} \]

region \( D \) is the semicircular region \( x^2 + y^2 \leq a^2, \ y \geq 0 \).

Calculate

\[ \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA = \oint_C \mathbf{F} \cdot d\mathbf{r} \]

Parameterize the curve \( C \)

\[ \begin{align*}
  y &= a \sin \theta, \quad 0 \leq \theta \leq \pi \quad \text{and} \\
  x &= a \cos \theta
\end{align*} \]

\[ \oint_C 2y \, dx + x \, dy = \int_0^\pi \left( 2a \sin \theta \cos \theta \, d\theta + \int_0^\pi a \cos \theta \, d\theta \right) \]

\[ = 2a^2 \left( \int_0^\pi \sin^2 \theta \, d\theta \right) + a^2 \left( \int_0^\pi \cos^2 \theta \, d\theta \right) \]

\[ = 2a^2 \left( \int_0^\pi (1 - \cos 2\theta) \, d\theta \right) + a^2 \left( \int_0^\pi (1 + \cos 2\theta) \, d\theta \right) \]

\[ = 2a^2 \left( \frac{\pi}{2} \right) + a^2 \left( \frac{\pi}{2} \right) = \frac{\pi a^2}{2} \]
\[ \iint_D (N_x - M_y) \, dA = \iint_D (1 - 2) \, dA = \iint_D -1 \, dA = \iint_D -dA \]

\[ N = x \quad N_x = 1 \]
\[ M = 2y \quad M_y = 2 \]

\[ x^2 + y^2 = 1 \quad \sqrt{x^2 - y^2} \]

\[ -\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} dx \, dy = -\int_{-a}^{a} \int_{0}^{\sqrt{a^2-x^2}} dy \, dx \]

Change of variables:

\[ = -\int_{0}^{\pi} \int_{0}^{\infty} r \, dr \, d\theta = -\int_{0}^{\pi} \left[ \frac{r^2}{2} \right]_0^\infty \, d\theta = -\int_{0}^{\pi} \frac{a^2}{2} \, d\theta \]

\[ = -\frac{\pi a^2}{2} \]
13) Show that if $D$ is a region for which Green's theorem applies, and $D$ is oriented so that $D$ is always on the left as we travel along $\partial D$, then the area of $D$ is given by either of the following surface integrals:

$$\text{Area of } D = \oint_{\partial D} x \, dy = -\oint_{\partial D} y \, dx$$

By Green's theorem, if the above conditions apply, then

$$\iint_{D} M \, dx + N \, dy = \iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA.$$

To find the area of $D$, we can find

$$\iint_{D} \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dA,$$

where the function $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1$.

(d) Double integral over the function 1 calculates the area.

Let $\frac{\partial N}{\partial x} = 1$ and $\frac{\partial M}{\partial y} = 0$, which satisfies the equation $\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} = 1 - 0 = 1$.

Then

$N$ can equal $x$ and $M$ can equal $0$.

(Note the "can equal" which means that there are many other possibilities of $N$ and $M$).

So by substituting into the left-hand side of Green's theorem,

$$\iint_{D} M \, dx + N \, dy = \iint_{D} 0 \, dx + 0 \, dy = \iint_{D} x \, dy$$

So the area of $D$ can be found by $\iint_{D} x \, dy$.
We can also let $\frac{M}{N} = 0$ and $\frac{M}{N} = 1$, which also satisfies the equation
\[\frac{\partial M}{\partial x} - \frac{\partial N}{\partial y} = 0 - (-1) = 1\]

Then,

$N$ can equal to $0$ and $M$ can equal to $-y$

So by substituting into the left-hand side of the Green's theorem,
\[\int_{0}^{y} \int_{0}^{x} N \, dx \, dy = \int_{0}^{x} y \, dx + \int_{0}^{y} x \, dy = \int_{0}^{x} -y \, dx = -\int_{0}^{y} x \, dx\]

so the area of $D$ can also be found by $\int_{0}^{y} x \, dx$

Thus,

area of $D = \int_{0}^{y} x \, dx = -\int_{0}^{y} x \, dx$
Use the divergence theorem to show that
\[ \iint_S \nabla \cdot \mathbf{F} \, dA = 0, \]
where
\[ \mathbf{F} = 2y\mathbf{i} - 3x\mathbf{j} \] and \( C \) is the circle \( x^2 + y^2 = 1 \).

By divergence theorem,
\[ \iint_S \nabla \cdot \mathbf{F} \, dA = \oint_C \mathbf{F} \cdot d\mathbf{s} \]

Since \( C \) is the circle \( x^2 + y^2 = 1 \),
The limit of the region \( D \) is:
\[ x: -1 \to 1 \]
\[ y: -\sqrt{1-x^2} \to \sqrt{1-x^2} \]
\[ dA = dx \, dy \]

Converting to polar coordinates, give
\[ dA = r \, dr \, d\theta \]
\[ r: 0 \to 1 \]
\[ \theta: 0 \to 2\pi \]

\[ \iint_D \nabla \cdot \mathbf{F} \, dA = \int_0^{2\pi} \int_0^1 \nabla \cdot \mathbf{F} \, r \, dr \, d\theta \]
\[ = \int_0^{2\pi} \int_0^1 \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) (2y, -3x) \, r \, dr \, d\theta \]
\[ = \int_0^{2\pi} \int_0^1 (0 - 0) \, r \, dr \, d\theta \]
\[ = 0 \]

\[ \Rightarrow \int_0^{2\pi} \int_0^1 (0 \cdot r \, dr \, d\theta = 0 \]
\[ \Rightarrow \int_0^{2\pi} \int_0^1 0 \, r \, dr \, d\theta = 0 \]
15.6) Now show $\int_C \mathbf{F} \cdot d\mathbf{s} = 0$ by direct computation of the line integral.

\[ x = \cos t, \quad y = \sin t, \quad 0 \leq t \leq 2\pi \]

\[ \mathbf{F} = (2y, -3x) \]

\[ (x, y) = 2xy - 3xy = -xy \]

Parameterize the curve

\[ \begin{align*}
  x &= \cos t \\
  y &= \sin t
\end{align*} \]

\[ ds = ||x'(t)|| dt \\
  x'(t) = (-\sin t, \cos t) \\
  ||x'(t)|| = \sqrt{\sin^2 t + \cos^2 t} = 1 \]

\[ ds = 1 \cdot dt \]

\[ \int_C \mathbf{F} \cdot \hat{n} ds = \int_C -xy ds = \int_0^{2\pi} -\cos t \sin t dt \]

\[ = -\int_0^{2\pi} \sin 2t dt = -\left[ \frac{\sin 2t}{2} \right]_0^{2\pi} = 0 \]

\[ = -\left( \frac{-\cos 4\pi + \cos 0}{4} \right) = -\left( -1 + 1 \right) = 0 \]
17) Let \( C \) be any simple, closed curve in the plane. Show that
\[
\oint_C y \, dx + x^3 \, dy = 0.
\]

By Green's theorem,
\[
\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy
\]
where \( D \) is the region bounded by the curve \( C \).

For the equation above:
\[
M = 3x^2y,
\]
\[
N = x^3.
\]

and
\[
\frac{\partial N}{\partial x} = 3x^2, \quad \frac{\partial M}{\partial y} = 3x^2.
\]

\[
\Rightarrow \oint_C 3x^2y \, dx + x^3 \, dy = \iint_D (3x^2 - 3x^2) \, dx \, dy = 0.
\]

20) Let \( f(x, y) \) be a function of class \( C^2 \) such that
\[
\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0.
\]

Show that if \( C \) is any closed curve in which
Green's theorem applies, then
\[
\oint_C \frac{\partial f}{\partial y} \, dx - \frac{\partial f}{\partial x} \, dy = 0.
\]

By Green's theorem,
\[
\oint_C M \, dx + N \, dy = \iint_D \left( \frac{\partial N}{\partial x} - \frac{\partial M}{\partial y} \right) \, dx \, dy
\]
where \( D \) is the region bounded by the curve \( C \).

for the equation above:
\[
M = \frac{\partial f}{\partial y}, \quad N = -\frac{\partial f}{\partial x}.
\]

\[
\Rightarrow \frac{\partial M}{\partial x} = \frac{\partial^2 f}{\partial x^2} \quad \text{and} \quad \frac{\partial N}{\partial y} = \frac{\partial^2 f}{\partial y^2}.
\]
by Green's theorem

\[ \int_C \frac{dy}{dx} \, dx - \frac{dx}{dy} \, dy = \iint_{D} \left( \frac{\partial^2 f}{\partial y^2} - \frac{\partial^2 f}{\partial x^2} \right) \, dx \, dy \]

\[ = \iint_{D} -\left( \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \right) \, dx \, dy \]

Let it be given that
\[ \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} = 0. \]

So by substitution
\[ \int_C \frac{dy}{dx} \, dx - \frac{dx}{dy} \, dy = \iint_{D} 0 \, dx \, dy = 0. \]