Multiplicative summability methods and the Stone-Čech compactification

By

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Introduction

A regular summability method $\varphi$ (for real sequences) is called multiplicative ($b$-multiplicative) if whenever $f, g$ are two (bounded) sequences summed by $\varphi$, then $\varphi(fg) = \varphi(f) \cdot \varphi(g)$. It is known [11], [17], p. 71 that the regular multiplicative matrix summability methods are submethods of the method $\varphi_I$ corresponding to the identity matrix $I$ (i.e., methods obtained from $I$ by deleting infinitely many of its rows). It was conjectured in [1] that every regular $b$-multiplicative matrix summability method is equivalent for bounded sequences to a submethod of $\varphi_I$. Below (3.3), we disprove this conjecture by exhibiting a countable decreasing chain of submethods of $\varphi_I$ such that the intersection of their bounded summability fields is not the bounded summability field of a submethod of $\varphi_I$, although it is the bounded summability field of a regular $b$-multiplicative matrix summability method.

More generally, we establish a one-one correspondence between the family of regular $b$-multiplicative summability methods and the family of all closed subsets of the complement $N'$ of $N$ in the Stone-Čech compactification $\beta N$ of the (discrete) space $N$ of positive integers. The bounded summability field of a submethod of $\varphi_I$ corresponds to an open and closed subset of $N'$. We are unable, however, to determine which closed subsets of $N'$ correspond to bounded summability fields of matrix methods of this type, although we are able to eliminate the finite subsets.

We also construct, with the aid of the continuum hypothesis, a simultaneously consistent family of submethods of $\varphi_I$ which together sum every bounded sequence of real numbers.

In section 1, we define the concepts needed in this paper, and state the necessary properties of $\beta N$. In section 2, we discuss consistent families of submethods of $\varphi_I$. Section 3 is devoted to regular $b$-multiplicative summability methods. In section 4, we compare our simultaneously consistent family of submethods of $\varphi_I$ to a pairwise $b$-consistent family of regular matrix summability methods which together sum every bounded real sequence that was constructed by GOFFMAN and PETERSEN [6]. Finally, in an appendix, we discuss what modifications are necessary to extend our results to complex sequences.

1) This paper was written while the author was an ALFRED P. SLOAN FELLOW.
I am indebted to C. Goffman for acquainting me with the literature on summability, and for many valuable conversations on this subject.

1. Definitions and preliminary remarks

If $X$ is a topological space, let $C(X)$ denote the set of all continuous real-valued functions on $X$, and let $C^*(X)$ denote the subset of all bounded elements of $C(X)$. Under the usual operations of pointwise addition and multiplication, these sets become algebras over the real field $R$. Indeed $C^*(X)$ becomes a Banach algebra if we let $\|f\| = \sup \{|f(x)| : x \in X\}$.

In particular, if $N$ is the (discrete) space of positive integers, then $C(N)$ is the algebra of all real sequences, and $C^*(N)$ is the Banach algebra of all bounded real sequences. We use $C_0(N)$ to denote the set of all convergent real sequences.

In this paper, we capitalize on certain known properties of the Stone-Cech compactification $\beta N$ of $N$. A self-contained discussion of it is given in [14]. Proofs of many of the following assertions are given there.

(1.1). The space $N$ is (homeomorphic to) a dense subspace of a compact (Hausdorff) space $\beta N$ such that every $f \in C^*(N)$ has a (unique) extension $\tilde{f} \in C(\beta N)$. The compact space $\beta N$, which is usually called the Stone-Cech compactification of $N$, is unique in the following sense. If $Y$ is a compact space containing $N$ as a dense subspace, and such that every $f \in C^*(N)$ has an extension $\tilde{f} \in C(Y)$, then there is a homeomorphism of $\beta N$ onto $Y$ keeping $N$ pointwise fixed [2], [14].

(1.2). If $Z \subset N$, let $Z^\beta$ denote the closure in $\beta N$ of $Z$. Each $Z^\beta$ is an open subset of $\beta N$.

To see this, consider the extension $\tilde{f} \in C(\beta N)$ of the bounded sequence $f$ such that $\tilde{f}(n) = 0$ if $n \in Z$, and $\tilde{f}(n) = 1$ otherwise. Thus the complement of $Z^\beta$ is closed, whence $Z^\beta$ is open [3], [14].

(1.3). If $Z_1$ and $Z_2$ are subsets of $N$, then $(Z_1 \cup Z_2)^\beta = Z_1^\beta \cap Z_2^\beta$. In particular, disjoint subsets of $N$ have disjoint closures in $\beta N$ [8], [10].

(1.4). An ultrafilter $\mathcal{U}$ on $N$ is a family of nonempty subsets of $N$ maximal with respect to the property of being closed under finite intersection. There is a one-one correspondence between the set of ultrafilters on $N$ and the points of $\beta N$, obtained as follows.

If $\mathcal{U}$ is an ultrafilter, then there is a unique point $\tilde{\mathcal{U}} \in \beta N$ common to every member of $\{Z^\beta : Z \in \mathcal{U}\}$. Conversely, if $\tilde{\mathcal{U}} \in \beta N$, then $\{Z \subset N : \tilde{\mathcal{U}} \in Z^\beta\}$ is an ultrafilter $\mathcal{U}$ such that $\tilde{\mathcal{U}} = \tilde{\mathcal{U}}$.

An ultrafilter $\mathcal{U}$ on $N$ is called fixed or free according as $\tilde{\mathcal{U}}(\mathcal{U})$ is in $N$ or not. Clearly, if $\mathcal{U}$ is fixed, then it consists of all subsets of $N$ containing $\tilde{\mathcal{U}}$, while if $\mathcal{U}$ is free, then every one of its elements is infinite [4], [14].

(1.5). We abbreviate the complement of $N$ in $\beta N$ by $N'$. Since $N$ is open in $\beta N$, the space $N'$ is compact. A subset $E$ of $N'$ is open and closed if and only if there is a $Z \subset N$ such that $E = Z^\beta \cap N'$. Indeed, $\{Z^\beta \cap N' : Z$ is an infinite subset of $N\}$ is a base for the open subsets of $N'$ [14].
(1.6). Let $\gamma R$ denote the two-point compactification of $R$. Every $f \in C(N)$ is a (continuous) mapping of $N$ into $\gamma R$. Hence by a theorem of Stone, $f$ has a continuous extension $\tilde{f}$ over $\beta N$ into $\gamma R$. Clearly if $f$ is bounded, then $\tilde{f} = f$ [see [4], and [6], Theorem 88].

(1.7). If every countable intersection of neighborhoods of a point $x$ of a compact space $X$ is a neighborhood of $x$, then $x$ is called a $P$-point of $X$. Equivalently, $x$ is a $P$-point of $X$ if and only if every $f \in C(X)$ is constant on a neighborhood of $x$. Every isolated point of $X$ is a $P$-point, and every $P$-point with a countable base of neighborhoods is isolated. It is easily verified that not every point of an infinite compact space can be a $P$-point [3].

Let $\mathcal{F} \in C(\beta N)$ be such that $\mathcal{F}(n) = n^{-1}$ for all $n \in N$. Clearly $\mathcal{F}$ vanishes precisely on $N'$, so no point of $N'$ can be a $P$-point of $\beta N$.

On the other hand, Walter Rudin has shown that if the continuum hypothesis is true, then the set of $P$-points of $N'$ is a dense subset of $N'$. Moreover, since $N'$ is compact and infinite, it must also contain non-$P$-points [14].

(1.8). A summability method $\mathcal{F}$ is a linear functional defined on a subspace of $C(N)$. The domain of $\mathcal{F}$ is called the summability field $\mathcal{F}(\mathcal{F})$ of $\mathcal{F}$, while $\mathcal{F}^* = C^*(N) \cap \mathcal{F}(\mathcal{F})$ is called the bounded summability field of $\mathcal{F}$. If $f \in \mathcal{F}(\mathcal{F})$, then $\mathcal{F}$ is said to sum $f$.

A summability method is called regular if $C_0(N) \subseteq \mathcal{F}(\mathcal{F})$, and $\mathcal{F}(f) = \lim_{n \to \infty} f(n)$ for all $f \in C_0(N)$.

If there is a countably infinite matrix $A = \langle a_{n,k} \rangle$ such that for every $f \in \mathcal{F}(\mathcal{F})$, $\mathcal{F}(f) = \lim_{n \to \infty} \sum_{k \in \mathbb{N}} a_{n,k} f(k)$, then $\mathcal{F}$ is called a matrix summability method, and $\mathcal{F}$ is sometimes denoted by $\mathcal{F}_A$. (Henceforth, we use "matrix" to denote "countably infinite matrix".)

It is well-known that no regular matrix summability method can sum all sequences of 0's and 1's (cf., e.g., [11]).

If $\mathcal{F}_A$ is a matrix summability method, $Z$ is an infinite subset of $N$, and the matrix $A(Z)$ is obtained from $A$ by deleting the $n$-th row of $A$ whenever $n \notin Z$, then $\mathcal{F}_A(Z)$ is called a submethod of $\mathcal{F}_A$. Clearly $\mathcal{F}_A(Z)$ is regular if $\mathcal{F}_A$ is regular [5].

Two summability methods $\mathcal{F}_1, \mathcal{F}_2$ are said to be consistent (resp. b-consistent) if whenever $f \in \mathcal{F}_1(\mathcal{F}_1) \cap \mathcal{F}_2(\mathcal{F}_2)$ [resp. $f \in \mathcal{F}_1(\mathcal{F}_2) \cap \mathcal{F}_2(\mathcal{F}_2)$], we have $\mathcal{F}_1(f) = \mathcal{F}_2(f)$.

A family $\{\mathcal{F}_n\}$ of summability methods is called pairwise consistent if every two elements of $\{\mathcal{F}_n\}$ are consistent. A family $\{\mathcal{F}_n\}$ of summability methods is called simultaneously consistent if whenever $\{\mathcal{F}_1, \ldots, \mathcal{F}_n\}$ is a finite subset of $\{\mathcal{F}_n\}$, $f_1, \ldots, f_n \in \mathcal{F}(\mathcal{F})$, and $f_1 + \cdots + f_n = 0$, then $\mathcal{F}_1(f_1) + \cdots + \mathcal{F}_n(f_n) = 0$. Clearly every simultaneously consistent family of summability methods is pairwise consistent, but the converse does not hold. In the obvious way, we define pairwise $b$-consistent and simultaneously $b$-consistent families [10].

If $\mathcal{F}_1, \mathcal{F}_2$ are regular matrix summability methods such that $\mathcal{F}_1(\mathcal{F}_2) \subseteq \mathcal{F}_2(\mathcal{F}_2)$, it is known that $\mathcal{F}_1$ and $\mathcal{F}_2$ are $b$-consistent [17, p. 67]. In this case, we say that $\mathcal{F}_1$ is weaker for bounded sequences than $\mathcal{F}_2$. 
2. Consistent families of submethods

Let \( I \) denote the identity matrix, so that \( q_I \) denotes ordinary convergence. Clearly \( f \in C_b(N) \) if and only if there is an \( a \in K \) such that \( f(\phi) = a \) for all \( \phi \in N' \), and \( q_I (f) = a \) in that event.

More generally, if \( Z = \{ s_1, s_2, \ldots, s_k, \ldots \} \) is an infinite subset of \( N \), then \( \lim_{k \to \infty} f(s_k) \) exists if and only if \( f \) is constant on \( \mathbb{Z}^* \cap N' \). Since \( q_{\{s_k\}} f = \lim_{k \to \infty} f(s_k) \), \( q_{\{s_k\}} f \) exists if and only if \( f \) is constant on \( \mathbb{Z}^* \cap N' \). Henceforth we abbreviate \( q_{\{s_k\}} \) by \( q_Z \).

Suppose that \( Z_1, Z_2 \) are infinite subsets of \( N \). It follows easily from (1.3) that \( \mathbb{Z}_1 ^* \cap N' = \mathbb{Z}_2 ^* \cap N' \) if and only if \( Z_1 \) and \( Z_2 \) differ only by a finite subset of \( N \). Moreover, as was noted in (1.5), every open and closed subset of \( N \) takes the form \( \mathbb{Z}^* \cap N' \) for some infinite subset \( Z \) of \( N \). Hence we have established

(2.1). The mapping \( \varphi_Z : \mathbb{Z}^* \cap N' \to \mathbb{Z}^* \cap N' \) is a one-to-one correspondence of the family of submethods of \( q_I \) onto the family of all open and closed subsets of \( N' \).

If \( Z_1, Z_2 \) are infinite subsets of \( N \), then clearly \( q_{Z_1} \) and \( q_{Z_2} \) are consistent if and only if \( Z_1 \cap Z_2 \) is infinite. For if \( Z_1 \cap Z_2 \) is finite, consider any sequence \( f \) such that \( f(n) = 0 \) if \( n \in Z_1 \), and \( f(n) = 1 \) if \( n \notin Z_2 \). Then \( q_{Z_1} f = 0 \), while \( q_{Z_2} f = 1 \). Conversely, if \( Z_1 \cap Z_2 \) is infinite, and both \( q_{Z_1} f \) and \( q_{Z_2} f \) exist, then \( q_{Z_1} f = q_{Z_1 \cap Z_2} f = q_{Z_2} f \).

Hence we have:

(2.2). If \( \{q_{Z_k}\} \) is a simultaneously consistent family of submethods of \( q_I \), then it is a subfamily of \( \{q_{Z_k} : Z \in \mathcal{F} \} \), where \( \mathcal{F} \) is a free ultrafilter on \( N \).

(2.3). If \( \mathcal{F} \) is a free ultrafilter on \( N \), then as was noted in (1.4), there is a unique point \( \phi \in N' \) common to all \( \mathbb{Z}^* \) such that \( Z \in \mathcal{F} \). Moreover, it is clear from (2.4) and the preceding comments that if \( Z \in \mathcal{F} \), and \( f \in \mathcal{F}(q_Z) \), then \( q_Z f = \hat{f}(\phi) \).

Suppose that \( \mathcal{F} \) is a free ultrafilter on \( N \). If \( Z \) is a subset of \( N \) that meets every element of \( \mathcal{F} \), then by definition of ultrafilter (1.4), \( Z \in \mathcal{F} \). So, we have

(2.4). If \( \mathcal{F} \) is a free ultrafilter on \( N \) then \( \{q_{Z_k} : Z \in \mathcal{F} \} \) is a simultaneously consistent family of submethods of \( q_I \), maximal with respect to the property of being pairwise consistent.

We have shown above that for any free ultrafilter \( \mathcal{F} \), and any \( Z \in \mathcal{F} \), that \( f \in \mathcal{F}(q_Z) \) if and only if \( f \) is constant on \( \mathbb{Z}^* \cap N' \). Hence \( f \in \mathcal{F}(q_Z) \) for some \( Z \in \mathcal{F} \) if and only if \( f \) is constant on a neighborhood in \( N' \) of \( \phi(\mathcal{F}) \).

Hence we have,

(2.5). If \( \mathcal{F} \) is a free ultrafilter on \( N \), then every \( f \in C^*(N) \) is summed by some \( q_Z \) with \( Z \in \mathcal{F} \) if and only if \( \phi(\mathcal{F}) \) is a P-point of \( N' \).

Next we state the main result of this section.

(2.6) Theorem. Suppose that the continuum hypothesis is true. If \( q_A \) is any regular matrix summability method, then there is a simultaneously consistent family of submethods of \( q_A \) such that every \( f \in C^*(N) \) is summed by at least one member of the family.
Proof. It suffices to prove the theorem in case $A = I$, for if \{q_{z_n}\} is such a family of submethods of $q_I$, then \{q_{z_n(x_n)}\} is the desired family of submethods of $q_I$.

The validity of the continuum hypothesis implies the existence of a $P$-point $\varphi$ of $N$ by (1.7). By \{1.4\}, there is a free ultrafilter $\mathcal{U}$ on $N$ with $\mathcal{P}(\mathcal{U}) = \varphi$.

By (2.4) and (2.5), \{q_{z^\mathcal{U}}: Z \in \mathcal{U}\} is the desired family.

(2.8). REMARK. The technique employed in the first paragraph of the proof shows that (2.2), (2.4), and (2.5) are valid if $q_I$ is replaced by any regular matrix summability method.

(2.7). REMARK. If $q_A$ is a regular row-finite matrix summability method, we will say that $f \in C(N)$ is summed by $q_A$ in the extended sense in case $q_A(f)$ exists, or if \[ \lim_{n \to \infty} \sum_{k=1}^{\infty} a_{n,k} f(k) = \pm \infty, \] where $A = (a_{n,k})$. It follows easily from (1.6) and the above that if $\mathcal{U}$ is a free ultrafilter such that $\mathcal{P}(\mathcal{U})$ is a $P$-point, then for every $f \in C(N)$, there is a $Z \in \mathcal{U}$ such that $f$ is summed in the extended sense by $q_A(Z)$.

3. Multiplicative summability methods

A summability method $q$ will be called multiplicative (resp. b-multiplicative) if whenever $f, g \in \mathcal{S}(q)$ (resp. $f, g \in \mathcal{S}^b(q)$), we have $q(fg) = q(f)q(g)$, and if $\mathcal{S}^b(q)$ is uniformly closed.

By a well known theorem of MAZUR and ORLICZ [11] (see also [17, p. 71]), a regular matrix summability method is multiplicative if and only if it is a submethod of $q_I$.

In [1], it was conjectured that every regular $b$-multiplicative matrix summability method is equivalent for bounded sequences to a submethod of $q_I$. Below we will show that this conjecture is false. First, however, we will study regular $b$-multiplicative summability methods in general.

Let $q$ denote any regular $b$-multiplicative summability method. Then $\mathcal{S}^b(q)$ is a uniformly closed subalgebra of the Banach algebra $C^b(N)$. It follows easily from (1.1), that $C^r(N)$ is isomorphic as a Banach algebra to $C(\beta N)$.

If $f, g \in \beta N$, let $f \cdot g$ denote the assertion that $f(\delta) = g(\delta)$ for all $f \in \mathcal{S}^b(q)$. It is clear that $\mathcal{S}^b(q)$ is an equivalence relation on $\beta N$, and that distinct points of $N$ lie in distinct equivalence classes. Let $X$ denote the quotient space of $N$ by the relation $\mathcal{R}$. It is clear that $X$ is a compact (Hausdorff) space [cf., e.g., [9, Chapter 3]]. Let $\delta$ denote the quotient mapping of $\beta N$ onto $X$. If $f \in \mathcal{S}^b(q)$, define the function $f'$ on $X$ by letting $f'(\delta(x)) = f(\delta(x))$ for all $x \in \beta N$. It is easily seen that the mapping $f \mapsto f'$ is an isomorphism of $\mathcal{S}^b(q)$ into the Banach algebra $C(X)$ (with the usual "sup" norm). Moreover the image of $\mathcal{S}^b(q)$ under this isomorphism is a uniformly closed subalgebra of $C(X)$ that contains the constant function, and such that if $x_1$ and $x_2$ are distinct points of $X$, there is an $f \in \mathcal{S}^b(q)$ such that $f'(x_1) \neq f'(x_2)$. Hence, by the celebrated Stone-Weierstrass approximation theorem [9, p. 244], the isomorphism is onto.

Clearly $q$ induces a multiplicative linear function $q'$ on the Banach algebra $C(X)$, so there is a point $x_0 \in X$ such that for any $f \in C(X)$, $q'(f) = f(x_0)$. 

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Hence $f \in \mathcal{S}^*(\varphi)$ if and only if $f$ is constant on the closed set $\delta^{-1}(x_0)$. We denote this closed set by $F(\varphi)$. Clearly $F(\varphi) \subset N'$.

Conversely, if $F$ is any closed subset of $N'$, then $\{f \in C^*(N) : f$ is constant on $F\}$ is the bounded summability field of a regular $b$-multiplicative summability method $\varphi$ such that $F = F(\varphi)$.

We summarize the above with the following theorem.

(3.4). Theorem. The mapping $\varphi \mapsto C^*(N)$ is a one-one correspondence between the family of restrictions to $C^*(N)$ of regular $b$-multiplicative summability methods and the family of all closed subsets of $N'$. Moreover, $f \in \mathcal{S}^*(\varphi)$ if and only if $f$ is constant on $F(\varphi)$.

If $\varphi_1$, $\varphi_2$ are two regular $b$-multiplicative summability methods such that $F(\varphi_1) \subset F(\varphi_2)$, then $\mathcal{S}^*(\varphi_2) \subset \mathcal{S}^*(\varphi_1)$. Conversely, if $\mathcal{S}^*(\varphi_2) \subset \mathcal{S}^*(\varphi_1)$, and $F(\varphi_1)$ has more than one point, then $F(\varphi_1) \subset F(\varphi_2)$. For, otherwise, there is a $\varphi_1$ in $F(\varphi_1)$, but not in $F(\varphi_2)$ and a $q \neq \varphi$ in $F(\varphi_2)$. Choose an $f \in C^*(N)$ such that $\tilde{f}(\varphi_1) = 0$ and $\tilde{f}(x) = 1$ if $x \in F(\varphi_2) \cup \{q\}$. Then $f$ is constant on $F(\varphi_2)$, but is not constant on $F(\varphi_1)$, so $f \notin \mathcal{S}^*(\varphi_1)$, and $f \notin \mathcal{S}^*(\varphi_1)$. Hence we have shown

(3.2). If $\varphi_1$, $\varphi_2$ are regular $b$-multiplicative summability methods such that $F(\varphi_1)$ contains more than one point, then $F(\varphi_1) \subset F(\varphi_2)$ if and only if $\mathcal{S}^*(\varphi_2) \subset \mathcal{S}^*(\varphi_1)$. Hence, if $\varphi$ is a regular $b$-multiplicative summability method that fails to sum a bounded divergent sequence, then the restriction of $\varphi$ to $C^*(N)$ is uniquely determined by $\mathcal{S}^*(\varphi)$.

As was noted in (1.8), the conclusion of the last part of (3.2) holds for any regular matrix summability method (whether $b$-multiplicative or not).

It seems natural to ask which closed subsets $F$ of $N'$ are such that there is a regular $b$-multiplicative matrix summability method such that $F = F(\varphi)$? Theorem 2.1 tells us that every open and closed subset $F$ of $N'$ has this property, and that every such $F$ determines a submethod of $\varphi_1$. We show next the existence of regular $b$-multiplicative matrix methods $\varphi$ such that $F(\varphi)$ is not open and closed.

(3.3). Example. Let $Z_1, Z_2, \ldots, Z_n, \ldots$ denote a sequence of subsets of $N$ such that for each $n \in N$, $Z_n \subset Z_{n+1}$ and there are infinitely many elements of $Z_{n+1}$ that are not in $Z_n$, and let $F_0 = Z_0 \cap N'$. It is clear that $F(\varphi_{Z_n}) = F_{n+1}$ for each $n \in N$, and that $F_n \subset F_{n+1}$. So, by (3.2), $\mathcal{S}^*(\varphi_{Z_n}) \subset \mathcal{S}^*(\varphi_{Z_{n+1}})$ for each $n \in N$.

A well-known theorem of Brouzono states that the intersection of a countable decreasing chain of bounded summability fields of regular matrix summability methods is the bounded summability field of a regular matrix summability method [17, p. 54]. Hence there is a regular matrix $A$ such that $\mathcal{S}^*(\varphi_A) = \bigcap_{n=1}^{\infty} \mathcal{S}^*(\varphi_{Z_n})$. Moreover it is clear that $\varphi_A$ is $b$-multiplicative, and it follows easily from (3.2) that $F(\varphi_A)$ is the closure in $N'$ of $\bigcup_{n=1}^{\infty} F_n$.

As we have already seen, $\varphi_A$ is equivalent for bounded sequences to a submethod of $\varphi_1$ if and only if $F(\varphi_A)$ is open and closed. If $F(\varphi_A) = F$ were open and closed, then by (1.5), there would be an infinite subset $Z$ of $N$ such that $F = Z \cap N'$. Since $F_n \subset F$ for each $n \in N$, $Z$ contains all but a finite number
of the elements of $Z_n$. Hence for each $n \in \mathbb{N}$, there are infinitely many elements of $Z_{n+1} \cap Z$ that are not in $Z_n \cap Z$; choose any such and call it $x_n$. Let $T$ denote the set obtained from $Z$ by deleting $\{x_n : n \in \mathbb{N}\}$. Now for each $n \in \mathbb{N}$, $T \cap Z_n$ differs from $Z_n$ by only a finite number of elements, so $(T \cap Z_n) \cap N = Z_n \cap N' = F_n$. It follows that $F \subseteq T \cap N'$. On the other hand, there are infinitely many elements of $Z$ that are not in $T$, so $T \cap N'$ is a proper subset of $Z \cap N' = F$. This contradiction shows that $F(q_A)$ is not open and closed, and hence that the regular $h$-multiplicative matrix summability method $q_A$ is not equivalent for bounded sequences to a submethod of $q_I$.

Actually, what we have given above is a proof of the known fact that the Boolean algebra of open and closed subsets of $N'$ is not $\sigma$-complete (see [7]).

If the reader has the patience to do so, he can construct an explicit matrix $A$ satisfying the conditions of Example 3.3 by interweaving with proper weighting factors the rows of the matrices $I(Z_n)$, $n = 1, 2, \ldots$, as in Brousseau's proof of the existence of a matrix $A$ such that $\mathcal{F}^*(q_A) = \bigcap_{n=1}^{\infty} \mathcal{F}^*(q_{Z_n})$ (see [17, p. 54]).

Note also that the argument given above cannot be used to conclude (falsely) that $q_A$ is a multiplicative summability method, even though each $q_{Z_n}$ is. For, a countable intersection of a strictly decreasing chain of summability fields of regular matrix summability methods need not be the summability field of a regular matrix summability method [17, p. 52].

The method $q_A$ is, however, weaker for bounded sequences than a submethod of $q_I$ (1.8).

By way of positive results in this direction, we have only (3.4) Theorem. If $q_A$ is a regular $h$-multiplicative matrix summability method, then $F(q_A)$ has power $2^\mathfrak{c}$ (where $\mathfrak{c}$ is the cardinal number of the continuum).

Proof. It suffices to prove that $F(q_A)$ is infinite, for every infinite closed subset of $\beta N$ has power $2^\mathfrak{c}$ [12].

Suppose that there is a positive integer $k$ such that $F(q_A)$ has precisely $k$ elements. If $k = 1$, then $\mathcal{F}^*(q_A) = C^*(N)$, which is impossible (4.8), so we may assume that $k \geq 1$. Suppose that $F(q_A) = \{\phi_1, \ldots, \phi_k\}$, so that $q_A$ sums $\{f \in C^*(N) : f(\phi_1) = \cdots = f(\phi_k)\}$. There is a $g \in C^*(N)$ such that $g(\phi_1) = 0$, while $g(\phi_2) = \cdots = g(\phi_k) = 1$, and there is a submethod $A_1 = A(Z)$ that sums $g$. Moreover, since $A(Z) = I(Z) A$, $\mathcal{F}^*(q_A) \subseteq \mathcal{F}^*(q_{A_1})$. Suppose that $h \in C^*(N)$, and $h(\phi_1) + h(\phi_2) = \cdots = h(\phi_k) = 1$. Let $h_1 = (h - h(\phi_1))(h(\phi_2) - h(\phi_1))^{-1}$. Then $h(\phi_1 - h)(\phi_1) = 0$, and $h(\phi_2 - h)(\phi_2) = \cdots = h(\phi_k - h)(\phi_k) = 1$, so $(h_1, g) \in \mathcal{F}^*(q_{A_1})$. Thus, $h_1 = h + (h_1 - g)$ is in $\mathcal{F}^*(q_{A_1})$, whence $h = (h(\phi_1) - h(\phi_2)) h_1 + h(\phi_k)$ is in $\mathcal{F}^*(q_{A_1})$. It follows that $\mathcal{F}^*(q_A)$ contains $\{f \in C^*(N) : f(\phi_1) = \cdots = f(\phi_k)\}$. Continuing in this way, we can construct a submethod $q_{A_1}$ of $q_A$ that sums all bounded sequences. This contradiction yields the theorem.

4. Some remarks on a paper of Goffman and Petersen

Let $\mathfrak{R}$ denote the class of all positive regular matrices such that for every $A \in \mathfrak{R}$, the number $\frac{1}{A}$ appears at least once in each row and each column of $A$.

In [6], Goffman and Petersen show that $\{q_A : A \in \mathfrak{R}\}$ is a pairwise $h$-con-
consistent family of regular matrix summability methods such that every bounded sequence \( f \) is summed by at least one member \( q_A \) of the family. Indeed, if \( U(f) \), \( l(f) \) denote respectively the limit superior and limit inferior of the sequence \( f \), then \( q_A(f) = \frac{U(f) + l(f)}{2} \). Thus, the family \( A \) defines a functional \( \Phi \) on \( C^*(N) \) whose value at \( f \) is \( \frac{U(f) + l(f)}{2} \).

Unfortunately, (as was pointed out to the author by C. Goffman), \( \Phi \) is not a summability method, i.e., \( \Phi \) is not linear. Equivalently, \( \{q_A: A \in \mathcal{A}\} \) is not simultaneously \( b \)-consistent. For, let \( f_1, f_2, f_3 \) denote respectively the sequences \( \{1, 0, 0, 0, 0, 0, 0, 0, 0, \ldots\}, \{0, 1, 0, 0, 0, 0, 0, 0, 0, \ldots\}, \{-1, -1, 0, -1, 0, -1, 0, -1, 0, \ldots\} \). Then \( \Phi(f_1 + f_2 + f_3) = \Phi(0) = 0 \), while \( \Phi(f_1) \Phi(f_2) \Phi(f_3) = \frac{1}{2} + \frac{1}{2} - \frac{1}{2} = \frac{1}{2} \). Moreover, as was pointed out by Anew in his review of [6] (Mathematical Reviews, vol. 17, p. 1200), the family \( \{q_A: A \in \mathcal{A}\} \) is not pairwise consistent (although, of course, it is pairwise \( b \)-consistent).

On the other hand, if \( \mathcal{F} \) is an ultrafilter on \( N \) such that \( \Phi(\mathcal{F}) \) is a \( P \)-point of \( N \). Then, as is shown in section 2, \( \{q_Z: Z \in \mathcal{F}\} \) is a simultaneously consistent family of regular matrix summability methods such that every bounded sequence is summed by some \( q_Z \). Moreover, the functional \( \Phi^\mathcal{F} \) defined on \( C^*(N) \) by this family [i.e., if \( f \in C^*(N) \), let \( \Phi^\mathcal{F}(f) = f(\Phi(\mathcal{F})) \)] is both linear and continuous.

Hence this latter family has a number of properties not possessed by the one constructed by Goffman and Petersen. On the other hand, the proof of its existence relies on the continuum hypothesis (2.6), and it is more difficult to visualize than that of Goffman and Petersen.

In [6], these authors pose the following problem. Can every pairwise \( b \)-consistent family of regular matrix summability methods be enlarged to a pairwise \( b \)-consistent family of regular matrix summability methods such that every bounded sequence is summed by some member of the enlarged family? In [13], Petersen answers this question in the affirmative for countable family of methods with bounded norm. (See [13] for the definition of norm.)

If \( \mathcal{F} \) is an ultrafilter on \( N \) such that \( \Phi(\mathcal{F}) \) is not a \( P \)-point, then by (2.5) the simultaneously consistent family \( \{q_A: Z \in \mathcal{F}\} \) does not sum every bounded sequence. (It is easy to show, however, that it will sum all bounded sequences with only a finite number of limit points.) This family impresses the author as a likely candidate for a negative answer to this problem, but he has been unable to discover a proof in either direction.

Appendix

All of the above discussion has been concerned with sequences of real numbers. It seems natural to ask what can be done with sequences of complex numbers.

All of the results of section 2 can be extended with trivial modification to handle complex sequences as well. For, every bounded complex-valued function on \( N \) has a (unique) continuous extension over \( \beta N \).
The results of section 3, however, rely on the Stone-Weierstrass approximation theorem. These too, can be extended to handle complex sequences if we require that whenever a function on \( N \) is summed by one of our summability methods \( \varphi \), then \( \varphi \) also sums its complex conjugate.

*Added October 9, 1958.* The author had conjectured that every regular \( \delta \)-multiplicative matrix summability method is weaker for bounded sequences than a submethod of \( \varphi \). At the International Congress of Mathematicians in Edinburgh, P. Erdős, relaying a message from K. Zeller, pointed out that this is false. See reference Zeller, §3(c) in [17]. The author will comment on this example and its ramifications in a future note.

**References**