Residue class fields of lattice-ordered algebras

by

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This paper is a continuation of [5], and is concerned with the structure of the residue class fields of the \( \Phi \)-algebras introduced and studied in that paper. These are archimedean lattice-ordered algebras with a multiplicative identity that is a weak order unit. The lattice-ordered ring \( C(Y) \) of all continuous real-valued functions on a topological space \( Y \) is a \( \Phi \)-algebra, and it is shown in [5] that every \( \Phi \)-algebra \( A \) is isomorphic to a ring of continuous functions from a compact space \( X \) into the two-point compactification of the real line \( R \) such that every \( f \in A \) is real-valued on an (open) dense subset of \( X \).

If \( A = C(Y) \), and \( M \) is a maximal \( l \)-ideal of \( A \), it is known that \( A/M \) is a real-closed field that is either the real field, or an \( \eta \)-set in its unique ordering. We show that for any uniformly closed \( \Phi \)-algebra \( A \), the residue-class fields are real-closed. This result seems to be new even for \( \Phi \)-algebras of real-valued functions. Stronger assumptions must be made to guarantee that if \( A/M \) is not the real field, then it is an \( \eta \)-set. We show that if \( A \) is closed under countable composition (i.e. if \( \{ f_n \} \) is a sequence of elements of \( A \), and \( g \in C(R^\infty) \), then there is an \( h \in A \) such that \( h(x) = g(f_1(x), ..., f_n(x), ...) \) whenever all of the \( f_n \) are real-valued), then \( A \) is closed under uniform convergence, and \( A/M \) is an \( \eta \)-set if it is not the real field. In fact, under this hypothesis, \( A \) is a homomorphic image of \( C(Y) \), for some topological space \( Y \).

It is shown also that every \( \Phi \)-algebra \( A \) is a homomorphic image of a \( \Phi \)-algebra \( B \) of real-valued functions; moreover, \( B \) can be chosen so that it is closed under countable composition, (finite) composition, uniform convergence, or bounded inversion, provided that \( A \) is.

An example is given of a uniformly closed \( \Phi \)-algebra \( A \) that is closed under (finite) composition, with a maximal \( l \)-ideal \( M \) such that \( A/M \) contains \( R \) properly, and has a countable cofinal subset. This serves to correct an error in [6].

The notation and terminology is that of [5]. An effort has been made to keep the exposition reasonably self-contained.

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1. Residue class fields of uniformly closed \(\Phi\)-algebras.

Recall from [5] that a \(\Phi\)-algebra \(A\) is said to be closed under bounded inversion provided \(1/a \in A\) whenever \(a \geq 1\) in \(A\).

1.1. Lemma. A \(\Phi\)-algebra \(A\) is closed under bounded inversion if and only if every maximal ideal of \(A\) is an \(l\)-ideal.

Proof. If \(a \geq 1\) in \(A\), then \(a\) is in no proper \(l\)-ideal of \(A\). Hence, if every maximal ideal of \(A\) is an \(l\)-ideal, then \(A\) is closed under bounded inversion.

For the converse, let \(M\) be a maximal ring ideal of \(A\) and suppose \(b \notin M, |a| \geq |b|\). Since \(A/M\) is a field, there is an \(x \in A\) such that \(bx + m = 1\). Squaring, we obtain \(b^2x^2 + m' = 1\), where \(m' = 2bxm + m^2 \in M\). Then, since \(b^2 \leq a^2\), and \(x^2 \geq 0\), we must have \(a^2x^2 + m' \geq 1\). If \(A\) is closed under bounded inversion, there is a \(z \in A\) such that \((a^2x^2 + m')z = 1\), so that \(a(ax^2) = 1 \pmod{M}\). Thus \(a \notin M\). Hence \(M\) is an \(l\)-ideal of \(A\).

As in [5], we say that \(A\) is uniformly closed if every Cauchy sequence of elements of \(A\) converges to an element of \(A\).

If \(\mathcal{X}\) is any compact space, let \(D(\mathcal{X})\) denote the set of all continuous functions defined on \(\mathcal{X}\) with values in the two point compactification \(\gamma R = [-\infty, +\infty]\) of the real line \(R\) that are real-valued on a dense (open) subset of \(\mathcal{X}\). If lattice operations are defined coordinatewise, then \(D(\mathcal{X})\) forms a lattice. Let \(f, g \in D(\mathcal{X})\). If there is an \(h \in D(\mathcal{X})\) such that \(h(x) = f(x) + g(x)\) whenever \(f(x)\) and \(g(x)\) are real, we write \(h = f + g\), and similarly for multiplication. In general, neither \(f + g\), nor \(f \cdot g\) is defined. It is true, however, that every \(\Phi\)-algebra \(A\) can be isomorphically represented in \(D(\mathcal{M}(A))\), where \(\mathcal{M}(A)\) is the space of maximal \(l\)-ideals of \(A\) with the Stone (= hull-kernel) topology. \(\mathcal{M}(A)\) is always a compact Hausdorff space ([5], Theorem 2.3). We will regard \(A\) as represented in this way whenever it is convenient to do so.

We will also utilize the following, proved in [5], 3.2 and 3.7.

1.2. The following properties of a \(\Phi\)-algebra \(A\) are equivalent.

(i) \(A\) is uniformly closed.

(ii) \(-A^*\) and \(C(\mathcal{M}(A))\) are isomorphic.

(iii) \(A\) is an order-convex subset of \(D(\mathcal{M}(A))\).

From (ii), it is evident that every uniformly closed \(\Phi\)-algebra is closed under bounded inversion.

If \(a \in A\), let \(\mathcal{R}(a) = \{x \in \mathcal{M}(A) : |a(x)| < \infty\}\), let \(\mathcal{L}(a) = \{x \in \mathcal{M}(A) : a(x) = 0\}\) and let \(\mathcal{H}(a) = \mathcal{M}(A) \sim \mathcal{R}(a)\). Finally, let \(\mathcal{R}(A) = \bigcap_{a \in A} \mathcal{R}(a)\). If \(\mathcal{R}(A)\) is dense in \(\mathcal{M}(A)\), then \(A\) is called an algebra of real-valued functions.

Let \(A\) be any \(\Phi\)-algebra, and let \(g \in C(\mathcal{H})\). If, for every \(f_1, \ldots, f_n \in A\), there is an \(h \in A\) such that \(h(x) = g(f_1(x), \ldots, f_n(x))\), whenever \(x \in \bigcap_{i=1}^n \mathcal{R}(f_i)\),
we say that $A$ is closed under composition with $g$, or that $A$ admits $g$. Evidently $h$ is unique; we shall write $h = g(f_1, \ldots, f_n)$. Every $\Phi$-algebra admits the constant functions and the projection functions $p_i$, where $p_i(\lambda_1, \ldots, \lambda_n) = \lambda_i$ ($i = 1, 2, \ldots, n$).

We let $F(A, n)$ denote the family of all $g \in C(R^n)$ that $A$ admits. It is easily verified that $F(A, n)$ is a $\Phi$-algebra if operations are defined in the usual coordinatewise fashion.

If $A$ is uniformly closed, so is $F(A, n)$. For if $\{g_i\}$ is a Cauchy sequence in $F(A, n)$, then it converges to some $g \in C(R^n)$. If $f_1, \ldots, f_n \in A$, then $\{g_i(f_1, \ldots, f_n)\}$ is a Cauchy sequence of elements of $A$ whose limit must be $g(f_1, \ldots, f_n)$.

Let $A$ be uniformly closed and let $p = \left(\sum_{i=1}^{n} p_i^{2}\right)^{1/2}$. Note that $\mathbb{R}^n \subset \mathcal{M}(F(A, n))$, and that $\mathcal{R}(p) = \mathbb{R}^n$. Hence by [5], Lemma 3.5, every $g \in C^*(\mathbb{R}^n)$ has a continuous extension over $\mathcal{M}(F(A, n))$, so $\mathcal{M}(F(A, n))$ and $\beta \mathbb{R}^n$ are homeomorphic. By 1.2 (iii), $F(A, n)$ is an order-convex sub-$\Phi$-algebra of $C(R^n)$. Thus, we have established

1.3. Lemma. If $A$ is a uniformly closed $\Phi$-algebra, then, for $n = 1, 2, \ldots, F(A, n)$ is a uniformly closed sub-$\Phi$-algebra of $C(R^n)$ containing all $g \in C(R^n)$ such that $|g| \leq \lambda(1 + p^2)^m$ for some $\lambda \in \mathbb{R}^+$, and some positive integer $m$.

Recall that a totally ordered field $F$ is called real-closed if every $a \in F^+$ has a square root and every polynomial of odd degree with coefficients in $F$ has a zero in $F$.

1.4. Theorem. If $A$ is a uniformly closed $\Phi$-algebra, and $M \in \mathcal{M}(A)$, then $A/M$ is a real-closed field.

Proof. Since $A$ is closed under bounded inversion, Lemma 1.1 shows that $A/M$ is a field. By [5] Theorem 3.8, every $a \in A^+$ has a square root, so we need only show that polynomials of odd degree with coefficients in $A/M$ have zeros.

Let $p_1(w) = \omega m^{m+1} + \lambda m^{m} + \ldots + \lambda_0$ denote a monic polynomial with real coefficients of positive degree. Let $r_1(\lambda), r_2(\lambda), \ldots, r_{m+1}(\lambda)$ denote the real parts of the complex zeros of $p_1(w)$ indexed so that $r_1(\lambda) \leq r_2(\lambda) \leq \ldots \leq r_{m+1}(\lambda)$. This serves to define $m+1$ real-valued functions on $R^{m+1}$. It is known that each of these functions is continuous ([4]). Moreover, by [9], p. 94, $|r_i(\lambda)| < 1 + |\lambda_0|^{1/2} + \ldots + |\lambda_m|^{1/2}$ for each $\lambda = (\lambda_0, \ldots, \lambda_m) \in R^{m+1}$, and $i = 1, \ldots, m+1$. Hence, by Lemma 1.3, $A$ is closed under composition with $r_i$.

Let $q(w) = \omega_{m+1} f_{m+1} m^{m} + \ldots + f_0$ denote a monic polynomial of odd degree with coefficients in $A$. By the above, $s_t = r_t(f_1, \ldots, f_{m+1}) \in A$. Since $q(w)$ has odd degree, for each $x \in \bigcap_{t=1}^{m+1} \mathcal{R}(f_t)$, there is an $i$ such that $q(s_i)(x) = 0$. 

Hence \( q(s_1)q(s_2)\ldots q(s_{m+1}) = 0 \). Since \( M \) is a prime ideal, there is an \( i_0 \) such that \( q(s_{i_0}) \in M \). Hence \( A/M \) is a real-closed field.

The argument just given enables us to reach the following slightly stronger conclusion. If \( A \) is uniformly closed \( \Phi \)-algebra, and \( P \) is a prime \( l \)-ideal of \( A \), then every positive element of \( A/P \) has a square root, and every monic polynomial of odd degree with coefficients in \( A/P \) has a zero in \( A/P \). Also, as we will show next, the assumption that \( P \) is an \( l \)-ideal is redundant.

1.5. Lemma. Every prime ideal \( P \) of a uniformly closed \( \Phi \)-algebra \( A \) is an \( l \)-ideal.

Proof. Since \( |c|^a = c^a \), we know that \( c \in P \) if and only if \( |c| \in P \). Thus, since \( |c| = (|c| \wedge 1)(|c| \vee 1) \), and since, by Lemma 1.2, \( A \) is closed under bounded inversion, \( c \in P \) if and only if \( |c| \wedge 1 \in P \).

Suppose now that \( |b| \leq |a| \), and \( a \in P \). Then \( |b| \wedge 1 \leq |a| \wedge 1 \in P \cap A^* \). But, by Lemma 1.2, \( A^* \) and \( C(\mathcal{M}(A)) \) are isomorphic, and by [3], Chapt. 14, every prime ideal of the latter is an \( l \)-ideal. So \( |b| \wedge 1 \in P \), whence \( b \in P \). Hence \( P \) is an \( l \)-ideal.

1.6. Remark. It is remarked in [3], Chapt. 13, that any totally ordered field containing \( R \) properly in which exponentiation of positive elements to real powers can be defined has degree of transcendency at least \( c \) over \( R \). It follows that if \( A \) is a uniformly closed \( \Phi \)-algebra, and \( M \in \mathcal{M}(A) \) is hyper-real, then \( A/M \) has degree of transcendency at least \( c \) over \( R \).

If \( S \) and \( T \) are subsets of a totally ordered set \( L \), and \( s < t \) whenever \( s \in S \) and \( t \in T \), we will write \( S < T \).

1.7. Theorem. Let \( P \) be a prime ideal of a uniformly closed \( \Phi \)-algebra \( A \). If \( S \) and \( T \) are countably infinite subsets of \( A/P \) such that \( S \) has no largest element, \( T \) has no smallest element, and \( S < T \), then there is an \( a \in A/P \) such that \( S < a < T \).

Proof. Since, by 1.5, \( P \) is a prime \( l \)-ideal, \( A/P \) is totally ordered, and by 1.2 ff., we may assume that \( 0 < S < T \leq 1 \). By Lemma 1.2, \( A^* \cong C(\mathcal{M}(A)) \). Kohls has shown that the conclusion follows in case \( A \cong C(\mathcal{Y}) \) for any space \( \mathcal{Y} \) ([8], Theorem 2.6). Since

\[
\frac{A^*}{P \cap A^*} = \frac{A^* + P}{P} = C/A/P,
\]

the conclusion holds in this case as well.

A totally ordered set \( L \) is called an \( \eta \)-set if whenever \( S \) and \( T \) are countable subsets of \( L \) such that \( S < T \), then there is an \( a \in L \) such that \( S < a < T \). In particular, an \( \eta \)-set has no countable cofinal or coinitial subset.

For any topological space \( \mathcal{Y} \), and any hyper-real maximal ideal \( M \) of \( C(\mathcal{Y}) \), it is known that \( C(\mathcal{Y})/M \) is an \( \eta \)-set. Example 1.9 below shows
strongly that no comparable conclusion holds for arbitrary uniformly closed $\mathcal{A}$-algebras.

Most of the remainder of the paper will be devoted to a discussion of the extra hypotheses needed to conclude that $A/M$ is an $\eta_1$-set.

A $\mathcal{A}$-algebra $A$ is said to be closed under (finite) composition if $P(A, n) = C(R^n)$ for $n = 1, 2, \ldots$; that is, if $A$ admits every $g \in C(R^n)$.

As in [5], $A$ is said to be closed under $l$-inversion if $\langle a \rangle = A$ whenever $\mathcal{L}(a) \subseteq \mathcal{A}(b)$ for some $b \in A$. (Recall that $\langle a \rangle$ is the smallest $l$-ideal of $A$ containing $a$.)

1.8. Lemma. Let $A$ be a $\mathcal{A}$-algebra.

(i) If $P(A, 2) = C(R^2)$ (in particular, if $A$ is closed under composition), then $A$ is closed under $l$-inversion.

(ii) If $A$ is closed under uniform convergence and $l$-inversion, then $A$ is closed under composition.

Proof. (i) Let $a, b \in A$, and suppose that $\mathcal{L}(a) \subseteq \mathcal{A}(b)$. Let $h = |a| \lor |b|$, let $B_h = \{ f \in \mathcal{H} : \mathcal{R}(f) \supseteq \mathcal{R}(h) \}$, and let $\mathcal{H} = \{ (a(x), b(x)) \in R^2 : x \in \mathcal{R}(h) \}$. If $(0, g) \in \mathcal{H}$, then there is a sequence $\{x_n\}$ of points of $\mathcal{R}(h)$ such that $a(x_n) \to 0$ and $b(x_n) \to g$. Since $\mathcal{H}(A)$ is compact, $\{x_n\}$ has a limit point $x \in \mathcal{H}(A)$. Clearly $a(x) = 0$, and $b(x) = g$, contrary to the assumption that $\mathcal{L}(a) \subseteq \mathcal{A}(b)$. Thus, the function $g$ defined on $\mathcal{H}$ by letting $g(f) = 1/p$ is continuous. By the Tietze extension theorem, it has an extension $\tilde{g} \in C(R^2)$. Since $P(A, 2) = C(R^2)$, this shows that $1/a \in A$.

(ii) Suppose that $f_1, \ldots, f_n \in A$, let $h = \max |f_1| \lor \cdots \lor |f_n|$, and let $B_h = \{ a \in A : \mathcal{R}(a) \supseteq \mathcal{R}(h) \}$. By [5], Theorem 5.8, since $A$ is closed under uniform convergence and $l$-inversion, $B_h$ and $C(\mathcal{R}(h))$ are isomorphic. Hence, for any $g \in C(R^n)$, $g(f_1, \ldots, f_n) \in A$ ($n = 1, 2, \ldots$). Thus, $A$ is closed under composition.

In [6], Theorem 1.28, Isbell states that if $A$ is an algebra of real-valued functions closed under uniform convergence and composition, and $M \in \mathcal{M}(A)$ is hyper-real, then $A/M$ is an $\eta_1$-set. While he establishes correctly the conclusion of Theorem 1.7 above, $A/M$ may have a countable cofinal subset, as is shown by the following. For $a \in A$, the image of $a$ in $A/M$ is denoted by $M(a)$.

1.9. Example. There exists a uniformly closed $\mathcal{A}$-algebra $A$, closed under composition, and a hyper-real $M \in \mathcal{M}(A)$ such that $A/M$ has a countable cofinal subset.

Proof. Let $\mathcal{E}$ denote the space of irrational numbers in $(0, 1)$ with its usual topology. Since $\beta \mathcal{E}$ is the largest compactification of $\mathcal{E}$, there is a continuous mapping $\pi$ of $\beta \mathcal{E}$ onto $[0, 1]$ keeping $\mathcal{E}$ pointwise fixed. Let $\mathcal{L}_0 = \pi^{-1}(0)$, and for $i = 1, 2, \ldots$, let $\mathcal{E}_i = \{1/p^j : p \in \mathcal{P}, j \in \mathcal{E}, \mathcal{E}_i \subset (0, 1) \}$ a prime; $j$ a positive integer, $j \in \mathcal{E}_i$, let $\mathcal{L}_i = \mathcal{L}_0 \cup \mathcal{E}_i$, and let $\mathcal{Y}_i = \beta \mathcal{E} \sim \mathcal{L}_i$. 
Observe that $\mathcal{D} \subseteq Y_{i+1} \subseteq Y_i$ for $i = 1, 2, \ldots$, and let $A_i = \{f \in D(\mathcal{Y}) : \mathcal{R}(f) \subseteq Y_i\}$. Since $Y_i$ contains $\mathcal{D}$, it is $C^*$-imbedded in $\mathcal{Y}$, so $A_i$ and $C(Y_i)$ are isomorphic. Finally, let $A = \bigcup_{i=1}^{\infty} A_i$.

If $\{f_n\}$ is a Cauchy sequence of elements of $A$, then (as is noted in [5], 3.1) $\mathcal{R}(f_n) = \mathcal{R}(f_{n+1}) = \cdots$ for all but finitely many of the $f_n$. Thus we may assume that $f_n \subseteq A_i$ for some $i$, whence $f_n$ converges. Similarly, any finite number of elements of $A$ is contained in some $A_i$. Thus $A$ is closed under uniform convergence and $l$-inversion.

If every point of the compact space $\mathcal{X}$ had a neighborhood meeting only finitely many of the sets $\{x_i \sim x_i\}$, then $\mathcal{X}$ itself would have such a neighborhood. But every neighborhood of $0$ meets infinitely many of the sets $\{x_i \sim x_i\}$, so this cannot be the case. Hence, there is an $x \in \mathcal{X}$ such that every neighborhood of $x$ meets infinitely many of the sets $\{x_i \sim x_i\}$. By a suitable change of notation, we may assume that every neighborhood of $x$ meets all such sets.

Now each $x_i$ is the inverse image of a closed subset of a metrizable space, and hence is a closed $G_\delta$. Hence there is an $f_i \in A_i^*$ such that $\mathcal{N}(f_i) = x_i$. Now, $M_x(f_i)$ is greater than all the constant functions, so $M_x$ is hyper-real. If $g \in A$, there is an $i$ such that $\mathcal{N}(g) \subseteq x_i$. Suppose there were an $h \in M_x$ such that $g + h \supseteq f_i$. Then $\mathcal{H}(f_i) \subseteq \mathcal{H}(g) \cup \mathcal{H}(h)$, and hence $\mathcal{N}(h) \supseteq \mathcal{N}(f_i) \sim \mathcal{N}(g) \cup x_i$. But this latter set has $x$ as a limit point, contrary to the fact that $h \in M_x$. We conclude that $\{M_x(f_i) : i = 1, 2, \ldots\}$ is a countable cofinal subset of $A/M_x$.

2. $\mathcal{D}$-algebras closed under countable composition. The example of the last section motivates the consideration of a more restricted class of $\mathcal{D}$-algebras.

We designate a countable product of copies of $\mathcal{D}$ as $\mathcal{D}^\infty$.

Let $A$ be a $\mathcal{D}$-algebra, and suppose that for every $g \in C(\mathcal{D}^\infty)$, and every sequence $\{f_n : n = 1, 2, \ldots\}$ of elements of $A$, there is an $h \in A$ such that $h(x) = g(f_1(x), \ldots, f_n(x), \ldots)$ whenever $x \in \bigcup_{n=1}^{\infty} \mathcal{R}(f_n)$; we say that $A$ is closed under countable composition. By the Baire category theorem, $\bigcup_{n=1}^{\infty} \mathcal{R}(f_n)$ is dense in $\mathcal{H}(A)$, so $h$ is unique. We denote it by $g(f_1, f_2, \ldots, f_n, \ldots)$.

Clearly, if $A$ is closed under countable composition, it is closed under composition, and hence, by Lemma 1.8, it is closed under $l$-inversion. This motivates the consideration of the following concept.

A $\mathcal{D}$-algebra $A$ is said to be closed under countable $l$-inversion provided that $\langle g \rangle = A$ for each $g \in A$ for which there is a sequence $\{f_n : n = 1, 2, \ldots\}$ of elements of $A$ such that $\mathcal{E}(g) \subseteq \bigcup_{n=1}^{\infty} \mathcal{N}(f_n)$. 
The relationship between these two latter concepts is given by

2.1. Theorem. A \( \mathcal{O} \)-algebra \( A \) is closed under countable composition if and only if it is uniformly closed and closed under countable \( l \)-inversion.

Proof of necessity. Suppose that \( A \) is closed under countable composition, and that \( Z(g) \subset \bigcup_{n=1}^{\infty} \mathcal{K}(f_n) \) for some \( g, i_1, \ldots, i_n, \ldots \) in \( A \).

Let \( g = f_s, \mathcal{Y} = \bigcap_{n=0}^{\infty} \mathcal{K}(f_n) \), and define \( \psi: \mathcal{Y} \to \mathbb{R}^{\infty} \) by letting \( \psi(y) = (f_s(y), f_1(y), \ldots, f_n(y), \ldots) \) for all \( y \in \mathcal{Y} \). Let \( \mathcal{H} \) denote the closure in \( \mathbb{R}^{\infty} \) of \( \psi[\mathcal{Y}] \). If \( x = (x_0, x_1, \ldots, x_n, \ldots) \in \mathcal{H} \), then \( x_0 \neq 0 \). For, otherwise there would be a sequence \( \{y_n\} \) of points of \( \mathcal{Y} \) such that \( \psi(y_n) \) converges to \( x \).

Since \( \mathcal{H}(A) \) is compact, \( \{y_n\} \) has an accumulation point in \( \mathcal{H}(A) \), which is a point of \( Z(g) \) not in \( \bigcup_{n=1}^{\infty} \mathcal{K}(f_n) \).

Hence the function \( r: \mathcal{H} \to \mathbb{R} \) defined by letting \( r(x_0, x_1, \ldots, x_n, \ldots) = 1/x_0 \) is well-defined and continuous. By the Tietze extension theorem ([7], p. 242), \( r \) has an extension \( s \in C(\mathbb{R}^{\infty}) \). Since \( A \) is closed under countable composition, \( s(f_s, f_1, \ldots, f_n, \ldots) \) is an element \( h \) of \( A \) such that \( gh = 1 \) on the dense subset \( \mathcal{Y} \) of \( \mathcal{H}(A) \). Thus \( h \) is the inverse of \( g \), whence \( \langle g \rangle = A \).

Suppose next that \( \{f_n\} \) is a Cauchy sequence of elements of \( A \); define \( \mathcal{Y} \) as above, define \( \psi: \mathcal{Y} \to \mathbb{R}^{\infty} \) by letting \( \psi(y) = (f_1(y), \ldots, f_n(y), \ldots) \) for all \( y \in \mathcal{Y} \), and let \( \mathcal{H} \) denote the closure of \( \psi[\mathcal{Y}] \) in \( \mathbb{R}^{\infty} \).

Since \( \{f_n\} \) is a Cauchy sequence, for every \( \varepsilon > 0 \) there is a positive integer \( m \) such that for every \( x = (x_0, x_1, \ldots, x_n, \ldots) \) of \( \psi[\mathcal{Y}] \), \( |x_p - x_q| < \varepsilon \) whenever \( p, q \geq m \). For any \( z \in \mathcal{H} \), if \( p, q \geq m \), then \( |z_p - z_q| \leq \varepsilon \). For, if not, for some such \( z, p, \) and \( q \), there is a \( \delta > 0 \) such that \( |z_p - z_q| = \varepsilon + 2\delta \). Then \( \{w \in \mathbb{R}^{\infty}: |w_p - z| < \delta \} \) is a neighborhood of \( z \) in \( \mathbb{R}^{\infty} \) that contains no point of \( \psi[\mathcal{Y}] \), contrary to the fact that \( z \in \mathcal{H} \). Hence, for each \( z \in \mathcal{H} \), \( \{z_n\} \) is a Cauchy sequence. Define \( s: \mathcal{H} \to \mathbb{R} \) by letting \( s(z) = \lim z_n \). It is easily verified that \( s \in C(\mathcal{H}) \). By the Tietze extension theorem, \( s \) has a continuous extension \( t \in C(\mathbb{R}^{\infty}) \). Since \( A \) is closed under countable composition, \( h = t(f_1, f_2, \ldots, f_n, \ldots) \in A \). Clearly \( \{f_n\} \) converges to \( h \). This completes the proof of the necessity.

Before proving the sufficiency, we prove two lemmas that are of independent interest.

2.2. Lemma. Let \( \mathcal{Y} \) be a subspace of a compact space \( \mathcal{X} \) such that for some countable family \( \mathcal{P} \) of closed subsets of \( \mathcal{X} \), for every pair of points \( p \in \mathcal{Y}, q \in \mathcal{X} \setminus \mathcal{Y} \) there is a set in \( \mathcal{P} \) containing \( p \) but not \( q \). Then \( \mathcal{Y} \) is a Lindelöf space.
Proof. Let \( \{ U_a : a \in \Gamma \} \) denote an open cover of \( \mathcal{Y} \). For each \( a \in \Gamma \), let \( R_a = \mathcal{Y} \sim U_a \), \( \sigma_a \) denote the closure of \( R_a \) in \( \mathcal{X} \), and let \( V_a = \mathcal{X} \sim \sigma_a \). Clearly \( V_a \cap \mathcal{Y} = U_a \), and the sets \( V_a \) cover \( \mathcal{X} \sim \mathcal{Y} \), where \( \mathcal{Y} = \bigcap \{ U_a : a \in \Gamma \} \). Clearly \( \mathcal{Y} \) is a compact subset of \( \mathcal{X} \sim \mathcal{Y} \).

Let \( \mathcal{F} \) denote the union of all those subsets of \( \mathcal{X} \) that are disjoint from \( \mathcal{Y} \), and are finite intersections of elements of \( \mathcal{Y} \). Then \( \mathcal{F} \) is \( \sigma \)-compact, and hence is a Lindelöf space. Thus, it suffices to show that \( \mathcal{Y} \subset \mathcal{F} \). But, for each \( p \in \mathcal{Y} \), by hypothesis, the intersection of all the elements of \( \mathcal{Y} \) containing \( p \) is disjoint from \( \mathcal{Y} \cap \mathcal{X} \sim \mathcal{Y} \). Hence some finite intersection of them is disjoint from \( \mathcal{Y} \). Hence \( \mathcal{Y} \subset \mathcal{F} \).

2.3. Corollary. Every subset of a compact space \( \mathcal{X} \) that is in the smallest family of subsets of \( \mathcal{X} \) containing the closed subsets and closed under countable union and intersection, is a Lindelöf space. In particular, for any \( \Phi \)-algebra \( \mathcal{A} \) and any sequence \( \{ f_n \} \) of elements of \( \mathcal{A} \), \( \bigcap_{n=1}^{\infty} \mathcal{R}(f_n) \) is a Lindelöf space.

Proof. Every closed subspace of \( \mathcal{X} \) satisfies the hypothesis of Lemma 2.2, so it suffices to show that if \( \mathcal{E}_n \) satisfies this latter condition with associated countable family of closed sets \( \mathcal{V}_n \) for \( n = 1, 2, \ldots \), then so does \( \mathcal{Y} = \bigcup_{n=1}^{\infty} \mathcal{E}_n \), and \( \mathcal{X} = \bigcap_{n=1}^{\infty} \mathcal{E}_n \). If \( p \in \mathcal{Y} \), \( q \in \mathcal{X} \sim \mathcal{Y} \), then \( p \in \mathcal{E}_n \) for some \( n \), and \( q \notin \mathcal{E}_n \), so there is an element of \( \mathcal{V}_n \) that contains \( p \) and not \( q \). Thus, \( \mathcal{Y} \) satisfies the hypothesis of Lemma 2.2 with associated countable family of closed sets \( \bigcup_{n=1}^{\infty} \mathcal{V}_n \). The proof for \( \mathcal{X} \) is similar.

2.4. Lemma. Let \( \mathcal{A} \) be a uniformly closed \( \Phi \)-algebra that is closed under countable \( l \)-inversion, let \( \{ f_n \} \) be a sequence of elements of \( \mathcal{A} \), let \( \mathcal{Y} = \bigcap_{n=1}^{\infty} \mathcal{R}(f_n) \), and let \( \mathcal{B} = \{ g \in \mathcal{A} : \mathcal{Y} \subset \mathcal{R}(g) \} \). Then \( \mathcal{B} \) and \( C(\mathcal{Y}) \) are isomorphic.

Proof. Clearly \( \mathcal{B} \) is a sub-\( \Phi \)-algebra of \( \mathcal{A} \). Since \( \mathcal{B} \subset \mathcal{A}^* \), \( \mathcal{H}(\mathcal{B}) = \mathcal{H}(\mathcal{A}) \), and since \( \mathcal{A} \) is uniformly closed, so is \( \mathcal{B} \). Since \( f_n \in \mathcal{B} \) for \( n = 1, 2, \ldots \), \( \mathcal{R}(\mathcal{B}) = \mathcal{Y} \). \( \mathcal{B} \) is also closed under inversion of elements without zeros in \( \mathcal{R}(\mathcal{B}) \). For, if \( Z(g) \cap \mathcal{R}(\mathcal{B}) = \emptyset \), then \( Z(g) \cap \bigcup_{n=1}^{\infty} \mathcal{V}(f_n) \), so, since \( \mathcal{A} \) is closed under countable \( l \)-inversion, \( 1/g \) is in \( \mathcal{A} \) and is real-valued on \( \mathcal{Y} \). By Corollary 2.3, \( \mathcal{Y} \) is a Lindelöf space, so by [5], Lemma 5.3, for every \( h \in C(\mathcal{Y}) \), there is a \( b \in C^*(\mathcal{H}(\mathcal{B})) \) such that \( h^{-1}(0) = Z(b) \cap \mathcal{Y} \). It follows from [5], Theorem 3.2 that \( \mathcal{B} \) and \( C(\mathcal{R}(\mathcal{B})) \) are isomorphic.

The proof of sufficiency for Theorem 2.1 is now easy in view of Lemma 2.4. If \( \{ f_n \} \) is any sequence of elements of a uniformly closed \( \Phi \)-algebra that is closed under countable \( l \)-inversion, then, by Lemma 2.4,
if \( \mathcal{Y} = \bigcap_{n=1}^{\infty} \mathcal{R}(f_n) \), then \( \mathcal{C}(\mathcal{Y}) \) is a subalgebra of \( \mathcal{A} \). So, for any \( g \in \mathcal{C}(\mathbb{R}^\omega) \), 
\( g(f_1, \ldots, f_n, \ldots) \) is in \( \mathcal{A} \).

Before returning to hyper-real residue class fields, we prove

2.5. Theorem. Every \( \Phi \)-algebra \( \mathcal{A} \) can be obtained as a homomorphic image of a \( \Phi \)-algebra \( \mathcal{B} \) of real-valued functions in such a way that if \( \mathcal{A} \) is uniformly closed, or closed under bounded inversion, or composition, or countable composition, then so is \( \mathcal{B} \).

Proof. \( \mathcal{B} \) will be defined as an algebra of continuous real-valued functions on \( \mathcal{M}(\Lambda) \times \mathcal{N} \) where \( \mathcal{N} \) is the discrete space of positive integers. Every element \( g \) of \( \mathcal{B} \) will be regarded as a sequence \( \{g_n\} \) of functions on \( \mathcal{M}(\Lambda) \), where \( g_n(p) = g(p, n) \), for all \( p \in \mathcal{M}(\Lambda) \). \( \mathcal{B} \) consists precisely of all those \( \{g_n\} \) which converge pointwise to an element of \( \mathcal{A} \) on a dense \( \mathcal{G}_\sigma \); i.e. those \( g \in \mathcal{C}(\mathcal{M}(\Lambda) \times \mathcal{N}) \) such that for some \( f \in \mathcal{A} \), and for some dense \( \mathcal{G}_\sigma \)-set \( \mathcal{Y} \subset \mathcal{M}(\Lambda) \), for each \( p \in \mathcal{Y} \), the sequence \( \{g(p, n)\} \) of real numbers converges in \( \gamma \mathbb{R} \subset [\gamma, +\infty) \) to \( f(p) \).

Since the intersection of two dense \( \mathcal{G}_\sigma \)-sets is dense, each \( g \in \mathcal{B} \) converges to a unique \( \lambda(g) \in \mathcal{A} \). Similarly, it is easily verified that \( \mathcal{B} \) is a \( \Phi \)-algebra, and that \( \lambda \) is a homomorphism of \( \mathcal{B} \) into \( \mathcal{A} \). Moreover, if \( f \in \mathcal{A} \), and \( g_n = (f \wedge n) \vee (\neg n) \) for \( n = 1, 2, \ldots \), then \( g_n(p) \) converges to \( f(p) \) for all \( p \in \mathcal{M}(\Lambda) \). Hence \( \lambda(g) = f \), so \( \lambda \) is a homomorphism of \( \mathcal{B} \) onto \( \mathcal{A} \).

Suppose that \( \mathcal{A} \) is closed under countable composition, that \( \{g_n\} \) is a sequence of elements of \( \mathcal{B} \), and that \( h \in \mathcal{C}(\mathbb{R}^\omega) \). For \( n = 1, 2, \ldots \), there is a dense \( \mathcal{G}_\sigma \)-set \( \mathcal{Y}_n \) in \( \mathcal{M}(\Lambda) \) such that for each \( p \in \mathcal{Y}_n \), \( g_n(p, m) \) converges to \( \lambda(g_n)(p) \). Let \( \mathcal{Y} \) denote the intersection of all the \( \mathcal{Y}_n \) and all \( \mathcal{R}(\lambda(g_n)) \), for \( n = 1, 2, \ldots \). Since \( \mathcal{M}(\Lambda) \) is compact, this countable intersection of dense \( \mathcal{G}_\sigma \)-sets is a dense \( \mathcal{G}_\sigma \)-set. Moreover, each \( \lambda(g_n) \) is real-valued on \( \mathcal{Y} \), and so is \( h(\lambda(g_1), ..., \lambda(g_n), ...) \in \mathcal{A} \). For each \( p \in \mathcal{Y} \), and for each \( n \), the real numbers \( g_n(p, m) \) converge to \( \lambda(g_n)(p) \). Then the points \( \{x_m\} \) of \( \mathbb{R}^\omega \), whose \( n \)-th coordinates are \( g_n(p, m) \), form a convergent sequence in \( \mathbb{R}^\omega \), whose limit \( \chi \) has as \( n \)-th coordinate \( \lambda(g_n)(p) \). Since \( h \) is continuous, \( h(x_m) \rightarrow h(\chi) \). Thus, the well-defined continuous function \( h(g_1, ..., g_n, ...) \in \mathcal{C}(\mathcal{M}(\Lambda) \times \mathcal{N}) \) is in \( \mathcal{B} \), since it converges pointwise on \( \mathcal{Y} \) to \( h(\lambda(g_1), ..., \lambda(g_n), ...) \). That is, \( \mathcal{B} \) is closed under countable composition.

Simplified versions of the preceding establish the remaining assertions.

In [1], 2.1, Corson and Isbell show that if an algebra \( \mathcal{A} \) of real-valued functions is closed under countable composition, then it is closed under composition for all higher cardinals. This fact may be used to establish the following.
2.6. Theorem. Every \( \Phi \)-algebra \( A \) closed under countable composition is a homomorphic image of \( C(Y) \) for some topological space \( Y \).

For, by Theorem 2.5, we may assume without loss of generality that \( A \) is an algebra of real-valued functions. Let \( Y \) denote the cartesian product of as many copies \( R \) of \( R \) as there are elements \( f \) of \( A \). Let \( e \) denote the mapping of \( \mathcal{R}(A) \) into \( Y \) such that the \( f \)-th coordinate \( e(x)_f \) of \( e(x) \) is \( f(x) \). Finally, let \( \tau g = g \cdot e \) for each \( g \in C(Y) \). By the result cited above, since \( A \) is closed under countable composition, and hence unlimited composition, \( \tau g \in A \) for all \( g \in C(Y) \). Clearly \( \tau \) is a homomorphism of \( C(Y) \) onto \( A \).

In [2], it is shown that if \( M \) is a hypereal maximal ideal of \( C(Y) \), for some topological space \( Y \), then \( C(Y)/M \) is an \( \eta \)-set. Hence, by Theorems 1.4 and 2.6, we have immediately

2.7. Corollary. If \( A \) is a \( \Phi \)-algebra closed under countable composition, and \( M \in \mathcal{M}(A) \) is hyper-real, then \( A/M \) is real-closed field that is an \( \eta \)-set.

2.8. Corollary. If \( A = D(\mathcal{M}(A)) \) is a \( \Phi \)-algebra, and \( M \in \mathcal{M}(A) \) is hyper-real, then \( A/M \) is a real-closed field that is an \( \eta \)-set.

Proof. By 2.1 and 2.7, it suffices to show that the \( \Phi \)-algebra \( A = D(\mathcal{M}(A)) \) is closed under countable \( l \)-inversion and uniform convergence. The latter follows immediately from Lemma 1.2. Let \( \{ f_n \} \) be a sequence of elements of \( A \) such that \( Z(g) \subset \bigcup_{n=1}^{\infty} \mathcal{M}(f_n) \). Then \( Z(g) \) is nowhere dense and \( g \) cannot be a divisor of zero. Thus, by [5], Theorem 3.9, \( 1/g \in A \).

In [2], it is shown that all real-closed \( \eta \)-fields of power \( \kappa \), are isomorphic, if \( \kappa > 0 \). It follows from Corollary 2.8 that, if \( \kappa = \kappa \), then all of the residue class fields of the \( \Phi \)-algebra of all Lebesgue measurable functions on \( R \), modulo the ideal of functions vanishing off sets of measure zero, are isomorphic. See [5], Corollary 3.10.

References


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