THE SPACE OF MINIMAL PRIME IDEALS
OF A COMMUTATIVE RING

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1. Introduction. Our interest in the space of minimal prime ideals of a commutative ring arises from the special features of this space in case the ring is \( C(X) \), the ring of all continuous real-valued functions on a topological space \( X \). For instance, if \( X \) is the one-point compactification of a countable discrete space \( N \), then the space of minimal prime ideals of \( C(X) \) is homeomorphic with \( \beta N \), the Stone-Čech compactification of \( N \). This was pointed out by C. W. Kohls [4], who initiated the study of minimal prime ideals in rings \( C(X) \).

It should be noted at the outset that a minimal prime ideal means a prime ideal that contains no smaller prime ideal. Thus, in an integral domain, the only minimal prime ideals is \((0)\). The following lemma provides a key tool for the study of such ideals in an arbitrary ring, which we always assume to be commutative.

1.1. Lemma. A proper prime ideal \( P \) of a ring \( A \) is minimal if and only if for each \( x \in P \) there exists \( a \in A \sim P \) such that \( ax \) is nilpotent.

It is easy to see that the stated condition is sufficient for minimality of \( P \). To prove necessity, one assumes that the condition is violated and uses a standard argument involving Zorn’s lemma to construct a prime ideal contained properly in \( P \).

2. The space \( \mathcal{P}(A) \). Let \( \mathcal{P} \), or more precisely \( \mathcal{P}(A) \), denote the set of all minimal prime ideals in a ring \( A \). The hull of a set \( S \subset A \) is

\[
\mathcal{h}(S) = \{ P \in \mathcal{P} : S \subset P \}.
\]

The kernel of a set \( \mathcal{S} \subset \mathcal{P} \) is

\[
\mathcal{k}(\mathcal{S}) = \mathcal{S} \cap \{ P : P \in \mathcal{P} \}.
\]

A topology is defined in \( \mathcal{P} \) by means of a closure operation: The closure of \( \mathcal{S} \) is the set \( \mathcal{h}(\mathcal{S}) \). Evidently, the family of sets \( \{ \mathcal{h}(a) : a \in A \} \) is a base for the closed sets in \( \mathcal{P} \).

2.1. Theorem. Let \( I \) be an ideal of \( A \). The mapping \( \tau \) defined by

\[
\tau(P) = P \cap I, \quad P \in \mathcal{h}(I)
\]

is a homeomorphism of the subspace \( \mathcal{h}(I) \) of \( \mathcal{P}(A) \) onto a subspace of \( \mathcal{P}(A/I) \).

The proof of this theorem is straightforward. In case \( I \) is the ideal of all nilpotents in \( A \), then \( \mathcal{h}(I) \) is all of \( \mathcal{P}(A) \) and the image under \( \tau \) is all of \( \mathcal{P}(A/I) \). Consequently,
we lose no generality in studying topological properties of \( \mathfrak{P}(A) \) if we assume that \( A \) has no nonzero nilpotent.

For any \( S \subseteq A \), we denote the annihilator of \( S \) by \( \mathcal{U}(S) \):

\[
\mathcal{U}(S) = \{ a \in A : as = 0 \text{ for all } s \in S \}.
\]

2.2. Theorem. For any element \( a \) in a ring without nonzero nilpotent, \( h(\mathcal{U}(a)) = \mathcal{U} \sim h(a) \). Thus, besides being closed by definition, the sets \( h(a) \) and \( h(\mathcal{U}(a)) \) are open.

This theorem follows directly from Lemma 1.1 which, for rings without nonzero nilpotent, simply says: a prime ideal \( P \) is minimal if and only if for all \( x \in P \), \( \mathcal{U}(x) \notin P \).

2.3. Corollary. \( \mathcal{U} \) is a Hausdorff space with a base of open-and-closed sets.

2.4. Corollary. An element in a ring without nonzero nilpotent belongs to some minimal prime ideal if and only if it is a divisor of zero.

Additional useful properties of annihilators and hulls of elements are:

2.5. Lemma. For all \( x, y, z \) in a ring without nonzero nilpotent,

(i) \( h(x) = h(\mathcal{U}(x)) \);
(ii) \( \mathcal{U}(xy) = \mathcal{U}(x) \cap \mathcal{U}(y) \);
(iii) \( \mathcal{U}(z) = \mathcal{U}(x) \cap \mathcal{U}(y) \) if and only if \( h(z) = h(x) \cap h(y) \);
(iv) \( \mathcal{U}(y) = \mathcal{U}(x) \) if and only if \( h(y) = h(\mathcal{U}(x)) \).

3. Compactness of \( \mathcal{U} \). A striking difference between the space \( \mathcal{U}(\mathfrak{A}) \) and more familiar spaces of ideals of a ring \( A \) (see, e. g. [1]) is that compactness of \( \mathcal{U}(\mathfrak{A}) \) is wholly unrelated to the presence of a unity in \( A \). Instead, compactness of \( \mathcal{U}(\mathfrak{A}) \) hinges upon the existence of a kind of complement in the sense that for each element \( x \) of \( A \) there shall exist \( x' \in A \) such that \( \mathcal{U}(x') = \mathcal{U}(x) \). This condition is sufficient for compactness; we have been able to prove that it is necessary only under the additional restriction stated next.

3.1. Definition. A ring \( A \) is said to satisfy the annihilator condition (or is an a. c. ring) if \( A \) has no nonzero nilpotent and for every \( x, y \in A \), there exists \( z \in A \) such that \( \mathcal{U}(z) = \mathcal{U}(x) \cap \mathcal{U}(y) \).

It is difficult to find a ring without nonzero nilpotent that is not a. c. Professor Harley Flanders provided the following example which, moreover, has a unity.

3.2. Example. Let \( K \) be an algebraically closed field and \( A = K \times K \times K \). In the ring \( K^A \) of all \( K \)-valued functions on \( A \), let \( F \) be the subring of functions that are 0 except on a finite subset of \( A \). Define \( x, y \in K^A \) by

\[
x(a, b, c) = a, \quad y(a, b, c) = b, \quad ((a, b, c) \in A),
\]

and let \( A \) be the smallest subring of \( K^A \) that contains \( F \), \( x \), \( y \), and the constants. In the ring \( A \), \( \mathcal{U}(x) \cap \mathcal{U}(y) \) is not the annihilator of any single element.

3.3. Theorem. The following conditions on a ring \( A \) without nonzero nilpotent are equivalent:

(a) \( A \) is a. c. and \( \mathcal{U}(A) \) is compact.
(b) For each \( x \in A \) there exists \( x' \in A \) such that \( \mathcal{U}(x') = \mathcal{U}(x) \).
Proof. (a) implies (b). For a given \( x \in A \), the existence of \( x' \) will follow from compactness of \( h(x) \) only. By Theorem 2.2,
\[
\bigcap \{ h(y) \cap h(x) : y \in \mathfrak{U}(x) \} = h(\mathfrak{U}(x)) \cap h(x) = \emptyset.
\]
Hence, there exist \( y_1, \ldots, y_n \in \mathfrak{U}(x) \) such that
\[
h(y_1) \cap \cdots \cap h(y_n) \cap h(x) = \emptyset,
\]
which implies \( h(y_1) \cap \cdots \cap h(y_n) = h(\mathfrak{U}(x)) \). Since \( A \) is a. c., there exists \( x' \in A \) such that \( \mathfrak{U}(x') = \mathfrak{U}(y_1) \cap \cdots \cap \mathfrak{U}(y_n) \). By Lemma 2.5, we have, \( h(x') = h(\mathfrak{U}(x)) \) and hence \( \mathfrak{U}(x') = \mathfrak{U}(x) \).

Now assume (b). That \( A \) is a. c. follows from Lemma 2.5 (ii), if we set \( z = (x'y')' \). To prove that \( \mathcal{P}(A) \) is compact, we need

3.4. Lemma. Let \( A \) satisfy (b). An ideal \( I \) in \( A \) is contained in some minimal prime ideal of \( A \) if (and only if) every member of \( I \) is a divisor of 0. In particular, a prime ideal in \( A \) is minimal if and only if each of its members is a divisor of 0. (Cf. Corollary 2.4).

To prove this lemma, we use Zorn's lemma to embed \( I \) in a prime ideal \( P \) each of whose members is a divisor of 0. Suppose that \( P \) is not minimal. Then there exists \( x \in P \) such that \( \mathfrak{U}(x) \subset P \). The element \( x' \) such that \( \mathfrak{U}(x') = \mathfrak{U}(x) \) clearly belongs to \( \mathfrak{U}(x) \), and so \( x + x' \in P \). Hence \( x + x' \in P \). But \( h(x') = h(\mathfrak{U}(x)) = \mathbb{P} \sim h(x) \), that is, every minimal prime ideal contains exactly one of \( x \) and \( x' \). Thus, \( h(x + x') = 0 \), so by Corollary 2.4, \( x + x' \) is not a divisor of 0. This contradicts \( x + x' \in P \).

To complete the proof of the theorem, let \( \{ h(x_n) \} \) be a family of basic closed sets in \( \mathcal{P} \) with empty intersection. If \( I \) is the ideal generated by \( \{ x_n \} \), then \( h(I) = \bigcap h(x_n) = \emptyset \). By the lemma, \( I \) contains a nondivisor of 0, say \( e \). Then there exist \( x_1, \ldots, x_n \) and \( a_1, \ldots, a_n \) in \( A \) such that \( e = \sum_{i=1}^n a_i x_n \), and we have
\[
\bigcap_{i=1}^n h(x_n) \subset h(e) = \emptyset.
\]

4. Countable compactness and basic disconnectivity of \( \mathcal{P} \).

4.1. Definition. A ring \( A \) is said to satisfy the countable annihilator condition (or is a c. a. c. ring) if \( A \) has no nonzero nilpotent and for each sequence \( \{ x_n \} \) in \( A \), there exists \( x \in A \) such that \( \mathfrak{U}(x) = \bigcap_{n=1}^\infty \mathfrak{U}(x_n) \).

Obviously, every c. a. c. ring is a. c. Any ring \( C(T) \) is a c. a. c. ring, as follows:
\[
x(t) = \sum_{n=1}^\infty 2^{-n} \min(|x_n(t)|, 1) \text{ for all } t \in T.
\]

4.2. Lemma. If \( B \) is a set in any ring \( A \), then in the space \( \mathcal{P}(A) \), the closure of \( \bigcup \{ h(\mathfrak{U}(b)) : b \in B \} \) is \( h(\mathfrak{U}(B)) \).

4.3. Corollary. If for every \( B \subset A \), there exists \( x \in A \) such that \( \mathfrak{U}(B) = \mathfrak{U}(x) \), then \( \mathcal{P}(A) \) is extremally disconnected.
The hypothesis of this corollary is an obvious strengthening of the countable annihilator condition. If only c. a. c. is assumed, the most that might be expected of \( \mathcal{P}(A) \) is that it be basically disconnected — a countable analogue of extremal disconnectivity — which is defined in [2] as follows: a space \( X \) is basically disconnected if every zero-set in \( X \) has a closed interior. However, c. a. c. by itself is not sufficient to make \( \mathcal{P}(A) \) basically disconnected. Our most general result in this direction is

4.4. Theorem. If \( A \) is a c. a. c. ring and \( \mathcal{P}(A) \) is locally compact, then \( \mathcal{P}(A) \) is basically disconnected.

Next, we have a property of all c. a. c. rings.

4.5. Theorem. If \( A \) is a c. a. c. ring, then \( \mathcal{P}(A) \) is countably compact.

A cluster point for an arbitrary sequence \( \{P_n\} \) in such a \( \mathcal{P}(A) \) is constructed as follows: Let \( \mathcal{U} \) be an ultrafilter on the integers with void intersection. For each \( x \in A \), let \( E(x) = \{n : x \in P_n\} \). Then the set \( \{x \in A : E(x) \in \mathcal{U}\} \) is a minimal prime ideal of \( A \) and is a cluster point of \( \{P_n\} \). The countable annihilator condition is used only in the proof of minimality.

5. Minimal prime ideals of \( \Phi \)-algebras. An archimedean lattice-ordered algebra over the real field with a unity element 1 that is also a weak order unit is called a \( \Phi \)-algebra. As was shown in [3], a \( \Phi \)-algebra is a natural generalization of a ring \( C(X) \), especially if one is concerned with spaces of ideals. If \( A \) is a \( \Phi \)-algebra, then the space \( \mathcal{M}(A) \) of all maximal \( I \)-ideals (an \( I \)-ideal \( I \) is a ring ideal such that \( b \in I \) and \( |a| \leq |b| \) implies \( a \in I \)) of \( A \) is a compact Hausdorff space. The set

\[
A^* = \{a \in A : |a| \leq \lambda \text{ for some real } \lambda\}
\]

of bounded elements of \( A \) is also a \( \Phi \)-algebra, and \( \mathcal{M}(A^*) \) is homeomorphic with \( \mathcal{M}(A) \). It is also true that \( \mathcal{P}(A^*) \) is homeomorphic with \( \mathcal{P}(A) \). Any \( \Phi \)-algebra \( A \) is isomorphic with a \( \Phi \)-algebra of extended real-valued, continuous functions on the space \( \mathcal{M}(A) \) that are real-valued on a dense subset of \( \mathcal{M}(A) \). The isomorphism carries \( A^* \) onto a subalgebra of \( C(\mathcal{M}(A)) \).

Each minimal prime ideal of \( A \) is contained in a unique maximal \( I \)-ideal. Thus, there is a mapping \( \iota \) of \( \mathcal{P}(A) \) onto \( \mathcal{M}(A) \), which is automatically continuous. We shall state the properties of this mapping for the case \( A = C(X) \), where \( X \) is a compact Hausdorff space; in view of the remarks of the preceding paragraph, this entails little loss of generality. Also, we use the well-known homeomorphism between \( X \) and \( \mathcal{M}(C(X)) \) and regard \( \iota \) as a mapping of \( \mathcal{P} \) onto \( X \) — for \( P \in \mathcal{P} \), \( \iota(P) \) is the unique point in \( X \) where all functions in \( P \) vanish.

5.1. Theorem. (a) \( \iota \) is one-one if and only if \( X \) is an F-space [2, p. 208].

(b) \( \iota \) is a homeomorphism if and only if \( X \) is basically disconnected.

(c) In case \( X \) is an F-space, \( \mathcal{P} \) is compact if and only if \( X \) is basically disconnected.

Any \( C(X) \) is an a. c. ring, in fact, a c. a. c. ring. Condition (b) of Theorem 3.3 is
therefore necessary and sufficient in order that \( \mathcal{P}(C(X)) \) be compact. In terms of the behavior of functions, this condition says:

(b') For each \( f \in C(X) \), there exists \( f' \in C(X) \) such that
\[
Z(f) \cup Z(f') = X \quad \text{and} \quad \text{int} [Z(f) \cap Z(f')] = \emptyset.
\]

(\( Z(f) \) denotes the zero-set of \( f \).) A more easily verified sufficient, but not necessary, condition for \( \mathcal{P}(C(X)) \) to be compact is given next.

5.2. Theorem. If, for every \( f \in C(X) \), the support of \( f \) is a zero-set, in particular if \( X \) is perfectly normal, then \( \mathcal{P}(C(X)) \) is compact.

We conclude with some illuminating examples.

5.3. A space \( \Gamma \) such that \( \mathcal{P}(C(\Gamma)) \) is locally compact, but not compact. Let \( W^* \) be the totally ordered space of ordinal numbers less than or equal to the first uncountable ordinal, \( \omega_1 \). \( \Gamma \) is the quotient space of \( W^* \) obtained by identifying \( \omega_0 \) with \( \omega_1 \).

A function \( f \in C(\Gamma) \) for which (b') fails may be defined as follows: \( f(n) = 1/n \) if \( 0 < n < \omega_0 \), \( f(\gamma) = 0 \) otherwise. Hence \( \mathcal{P}(C(\Gamma)) \) is not compact. It is locally compact, however.

5.4. A space \( X \) such that \( \mathcal{P}(C(X)) \) is compact but for which the hypothesis of Theorem 5.2 is not satisfied. Let \( N^* \) denote the totally ordered space of ordinals less than or equal to \( \omega_0 \). With \( \Gamma \) as in 5.3, \( X \) is the complement in \( \Gamma \times N^* \) of the set \( \{(m, n) : m < \omega_0, n < \omega_0\} \).

5.5. A space \( X \) such that no open set in \( \mathcal{P}(C(X)) \) has compact closure. Let \( X = \beta N \sim N \), where \( N \) is a countable discrete space. We do not know whether \( \mathcal{P}(C(X)) \) is basically disconnected.

References