On a class of regular rings that are elementary divisor rings

By

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1. Introduction. Recall that a ring $\mathcal{R}$ is said to be regular in the sense of von Neumann if for every $a \in \mathcal{R}$, there is an $x \in \mathcal{R}$ such that $axa = a$. If, in addition, $\mathcal{R}$ has an identity element 1, and there is such an $x$ that is a unit (= element with two-sided inverse), we shall call $\mathcal{R}$ a unit-regular ring. This class of rings was introduced by G. Ehrlich who showed that every semi-simple Artinian ring is unit-regular, as is every strongly regular ring [2, Theorems 1 and 3]. (A ring $\mathcal{R}$ is called strongly regular if for every $a \in \mathcal{R}$, there is an $x \in \mathcal{R}$ such that $a^2x = a$. In [4, Lemma 10], it was shown that every commutative regular ring is unit-regular, and their proof carries over to the strongly regular case as is, once one observes that one-sided ideals in a strongly regular ring are two-sided. See [1].) She notes also that the ring of all linear transformations on an infinite dimensional vector space is regular, but not unit-regular.

For any positive integer $n$, let $\mathcal{R}_n$ denote the ring of $n \times n$ matrices with entries in $\mathcal{R}$. We call $\mathcal{R}$ an elementary divisor ring if for every $A \in \mathcal{R}_n$, there are units $P$ and $Q$ in $\mathcal{R}_n$ such that $PAP$ is a diagonal matrix. (See [5].) In this note, we show that every unit-regular ring is an elementary divisor ring. From this it follows that if $\mathcal{R}$ is unit-regular (in particular if $\mathcal{R}$ is strongly regular), so is $\mathcal{R}_n$. (By making use of unpublished work of Kaplansky we note that this latter result is not really new.) We show also that a necessary condition for unit-regularity given in [2, Theorem 6] is not sufficient. We close with some remarks and problems.

2. The main results. Henceforth, unless the contrary is stated explicitly, we will use the word "ring" to abbreviate the phrase "ring with identity element 1".

The following lemma will be used frequently in the sequel.

**Lemma 1.** If $a$ is an element of the ring $\mathcal{R}$, then the following statements are equivalent.

(i) There is a unit $u \in \mathcal{R}$ such that $aua = a$.

(ii) There is a unit $u \in \mathcal{R}$ such that $au$ and $ua$ are idempotents.

(iii) There is a unit $u \in \mathcal{R}$ such that either $au$ or $ua$ is idempotent.

(iv) There are units $p$ and $q$ in $\mathcal{R}$ such that $pqa$ is idempotent.

**Proof.** If (i) holds, then $(au)^2 = (aua)u = au$ and $(ua)^2 = u(aua) = ua$, so (ii) holds.

Clearly (ii) implies (iii), and (iii) implies (iv) since $\mathcal{R}$ has an identity element.
If (iv) holds, then \((paq)(paq) = paq\). By pre-multiplying both sides of this equation by \(p^{-1}\) and by post-multiplying by \(q^{-1}\), we see that
\[
a(qp) a = a.
\]
So \(u = qp\) is the unit needed to see that (i) holds. This completes the proof of the lemma.

This lemma is used first to obtain a necessary condition for a regular ring to be unit-regular.

**Proposition 2.** If \(a\) is an element of a ring \(R\) that satisfies any one of the conditions of Lemma 1, and \(ab = 1\) for some \(b \in R\), then \(ba = 1\). That is, one-sided inverses in a unit-regular ring are two-sided.

**Proof.** If \(u\) is a unit of \(R\) such that \(aua = a\), then \(au = (aua)b = ab = 1\), so \(u = 1\), whence \(b = u\) is the (two-sided) inverse of \(a\).

Let \(\mathcal{L}(V)\) denote the ring of all linear transformations of a vector space \(V\) over a division ring \(R\). That \(\mathcal{L}(V)\) is regular is well known; see, for example [8, p. 131]. Ehrlich shows that if \(V\) is infinite dimensional, then \(\mathcal{L}(V)\) fails to be unit-regular by showing that it violates the conclusion of Proposition 2, and proves that it is unit-regular in case \(V\) has finite dimension \(n\) by verifying that \(\mathcal{D}_n\) satisfies condition (iii) of Lemma 1. We will make use of Lemma 1 in proving the main result of this note; namely:

**Theorem 3.** Every unit-regular ring \(R\) is an elementary divisor ring.

In [5], Kaplansky calls a ring \(R\) right Hermite if for every positive integer \(n\), and \(A \in \mathbb{R}_n\), there is a non-singular \(Q\) in \(\mathbb{R}_n\) such that \(AQ\) is (upper) triangular. In [5, p. 465, Theorem 3.5, and Theorem 5.1] it is shown that \(R\) is a right Hermite ring (resp., elementary divisor ring) if for every \(A \in \mathcal{R}_2\), there is a non-singular matrix \(Q\) such that \(AQ\) is a lower triangular (resp., there are non-singular matrices \(P\) and \(Q\) such that \(PAQ\) is a diagonal matrix).

To prove Theorem 3, we need several lemmas. First we note that:

(1) \(If A \in \mathcal{R}_n\), all the entries of \(A\) commute with each other, and \(\text{det} \ A = 1\), then \(A\) is non-singular.

For, then we may regard \(A\) as a matrix over a commutative ring.

**Lemma 4.** Every unit-regular ring \(R\) is a right Hermite ring.

By the above, it suffices to find for each \(A = \begin{bmatrix} a & b \\ c & d \end{bmatrix}\) in \(\mathcal{R}_2\) a non-singular \(Q \in \mathcal{R}_2\) such that \(AQ\) is upper triangular.

By Lemma 1, there are units \(u, v\) in \(R\) such that \(f = eu\) and \(c = dv\) are idempotents. Clearly \(Q_1 = \text{diag}(u, v)\) is non-singular, and
\[
AQ_1 = \begin{bmatrix} au & bv \\ f & c \end{bmatrix} = A_1.
\]

If \(Q_2 = \begin{bmatrix} 1 & f - 1 \\ -f & 1 \end{bmatrix}\), then \(Q_2\) is non-singular by (1), and
\[ A_1 Q_2 = \begin{pmatrix} a_1 & b_1 \\ (1 - e) f & e \end{pmatrix} = A_2 \]

where \( a_1 = a u - b v f \) and \( b_1 = a u (f - 1) + b v \).

By Lemma 1,

(2) there is a unit \( w \in S \) such that \( f_1 = (1 - e) f w \) is idempotent.

Hence \( Q_3 = \text{diag}(w, 1) \) is non-singular and

\[ A_2 Q_3 = \begin{pmatrix} a_1 w & b_1 \\ f_1 & e \end{pmatrix} = A_3. \]

Note next that

(3) \( e f_1 = 0 \) and if \( g = f_1 (1 - e) \), then \( eg = ge = 0 \)

(see [7, p. 68]) and

(4) \( (1 - f_1 e) (1 + f_1 e) = (1 + f_1 e) (1 - f_1 e) = 1 \).

Thus \( Q_4 = \text{diag}(1 - f_1 e, 1 + f_1 e) \) is non-singular and

\[ A_3 Q_4 = \begin{pmatrix} a_2 & b_2 \\ g & e \end{pmatrix} = A_4 \]

where \( a_2 = a_1 w (1 - f_1 e) \) and \( b_2 = b_1 (1 + f_1 e) \).

Finally, if we let

\[ Q_5 = \begin{pmatrix} 1 - g & 1 \\ -g & 1 \end{pmatrix} \]

then \( Q_5 \) is non-singular by (1), and

(5) \[ A_4 Q_5 = \begin{pmatrix} a_3 & b_3 \\ 0 & e + g \end{pmatrix} = A Q \]

where \( a_3 = a_2 (1 - g) - b_2 g \), \( b_3 = a_2 + b_2 \), and \( Q = Q_1 Q_2 Q_3 Q_4 Q_5 \) is non-singular.

This completes the proof of Lemma 4.

**Lemma 5.** If \( S \) is unit-regular, and \( A \in S_2 \), then there are non-singular matrices \( P, Q \) such that \( PAQ \) is upper triangular and has idempotent entries.

**Proof.** By (3), (5), and Lemma 4, we may assume that

\[ A = \begin{pmatrix} a & b \\ 0 & e \end{pmatrix} \]

where \( e \) is an idempotent. By Lemma 1, there are units \( u, v \) in \( S \) such that \( ub \) and \( (u a) v \) are idempotents. Then \( P = \text{diag}(u, 1) \) and \( Q = \text{diag}(v, 1) \) are non-singular, and

\[ PAQ = \begin{pmatrix} (u a) v & ub \\ 0 & e \end{pmatrix} \]

has idempotent entries.
Lemma 6. If $R$ is unit-regular, and $A \in R_2$, then there are non-singular matrices $P$, $Q$ such that
\[ PAQ = \begin{bmatrix} e & g \\ 0 & h \end{bmatrix} \]
where $e$, $g$, and $h$ are idempotents, and $eg = ge = 0$.

Proof. By Lemma 5, we may assume that
\[ A = \begin{bmatrix} e & f \\ 0 & k \end{bmatrix} \]
where $e$, $f$, and $k$ are idempotents.

By (1)
\[ Q_1 = \begin{bmatrix} 1 & -f \\ 0 & 1 \end{bmatrix} \]
is non-singular, and
\[ AQ_1 = \begin{bmatrix} e & (1 - e) f \\ 0 & k \end{bmatrix} = A_1. \]

If we choose $w$ as in (2) and let $f_1 = (1 - e)f/w$, then $P_1 = \text{diag}(1, w^{-1})$ and $Q_2 = \text{diag}(1, w)$ are non-singular, and
\[ P_1 A_1 Q_2 = \begin{bmatrix} e & f_1 \\ 0 & w^{-1} k w \end{bmatrix} = A_2. \]

Finally, by (4) we know that $P_3 = \text{diag}(1, 1 + f_1 e)$ and $Q_3 = \text{diag}(1 + f_1 e, 1 - f_1 e)$ are non-singular, and
\[ P_2 A_2 Q_3 = \begin{bmatrix} e & g \\ 0 & h \end{bmatrix} = PAQ, \]
where $g = f_1 (1 - e)$, $h = (1 - f_1 e)^{-1} w^{-1} k w (1 - f_1 e)$, $P = P_2 P_1$, and $Q = Q_2 Q_3$.

By (3), $PAQ$ has the desired properties.

By Lemmas 4 through 6 and the remarks made proceeding them, to complete the proof of Theorem 3 we need only show that if
\[ A = \begin{bmatrix} e & g \\ 0 & h \end{bmatrix} \]
where $e$, $g$, and $h$ are idempotents, and $eg = ge = 0$, then there are non-singular matrices $P$, $Q$ such that $PAQ$ is a diagonal matrix. Now if
\[ Q = \begin{bmatrix} 1 - g & e - 1 \\ g & 1 - g \end{bmatrix} \]
then $Q$ is non-singular by (1), and
\[ AQ = \begin{bmatrix} e + g & 0 \\ hg & h(1 - g) \end{bmatrix}. \]
Also, by (1), the matrix

\[ P = \begin{bmatrix} 1 & 0 \\ -h & 1 \end{bmatrix} \]

is non-singular, and

\[ PAQ = \begin{bmatrix} e + g & 0 \\ 0 & h(1 - g) \end{bmatrix} \]

By what we have just shown, if \( R \) is unit-regular and \( A \in R_n \), there are non-singular matrices \( P \) and \( Q \) such that \( PAQ = \text{diag}(a_1, a_2, \ldots, a_n) \). By Lemma 1, there are units \( u_i \) such that \( a_i u_i \) is idempotent for \( i = 1, 2, \ldots, n \). Hence if \( U = \text{diag}(u_1, u_2, \ldots, u_n) \) then \( (PAQ)U \) is idempotent, so \( R_n \) is unit-regular by Lemma 1. Since by [2, Theorem 3], every strongly regular ring is unit-regular, we have established:

**Corollary 7.** If \( R \) is unit-regular (in particular if it is strongly regular), then so is \( R_n \).

Next, we take advantage of some unpublished results of I. Kaplansky to show that Corollary 7 is due, in essence, to Vasershtein [9].

**Proposition 8 (Kaplansky).** A regular ring \( R \) is unit-regular if and only if

(1) \( aR + bR = R \) implies there is a \( t \) in \( R \) such that \( a + bt \) is a unit.

**Proof.** Suppose first that \( R \) is unit-regular and that \( aR + bR = R \). By Lemma 1, there are idempotents \( e = au \) and \( f = by \) such that \( u \) is a unit. Then

\[ aR + bR = eR + fR = eR + (1 - e)fR. \]

Since \( R \) is regular, there is a \( w \in R \) such that \( (1 - e)f w (1 - e) = (1 - e)f. \)

Let \( g = (1 - e)f w (1 - e) \). It is easily verified that

\[ g^2 = g \quad \text{and} \quad eg = ge = 0 \]

and

\[ eR + fR = eR + gR = R \]

see (3) above.

Hence there are \( \alpha, \beta \) in \( R \) such that

\[ e\alpha + g\beta = 1. \]

Using (6), we have immediately that \( e\alpha = e \) and \( g\beta = g \), so

\[ e + (1 - e)f w (1 - e) = 1 \]

or

\[ au[1 - ef w (1 - e)] + by w (1 - e) = 1. \]

Multiplying on the right by \( y = [1 + ef w (1 - e)] u^{-1} \), we see that

\[ a + by w (1 - e) y = y \]

has a (two-sided) inverse, so (1) holds.
Assume next that (*) holds. If \( axa = a \), then
\[
aR + (1 - ax) R = R
\]
so there is a \( t \in R \) and a unit \( u \in R \) such that
\[
[a + (1 - ax)t] u = 1.
\]
Multiplying on the left by \( ax \) and on the right by \( a \) yields
\[
aua = a .
\]
This completes the proof of Proposition 8.

In [9], Vasershtein proves that if \( R \) is any (not necessarily regular) ring satisfying (*), then so does \( R^n \). Hence Corollary 7 should be credited to him (and Kaplansky).

We will call \( R \) a dependent ring if, for every \( a, b \in R \), there are \( s, t \in R \), not both zero, such that
\[
sa + tb = 0 .
\]

In [2, Theorem 6] Ehrlich shows that every unit-regular ring is dependent. We conclude this section by showing that not all dependent regular rings are unit-regular.

First, we prove:

**Theorem 9.** A regular ring \( R \) is dependent if and only if whenever \( a, a', b, b' \) are elements of \( R \) such that
\[
aa' = bb' = 1 ,
\]
then
\[a(1 - b'b) \text{ fails to have a right inverse.}
\]

**Proof.** Suppose first that for every \( a, b \) in \( R \) there are \( s, t \) in \( R \), not both zero, such that
\[
sa + tb = 0 .
\]
Hence
\[
sab' + t = 0
\]
so
\[
t = -sab'b .
\]
By (8), \( s \neq 0 \), and using (7), we see that
\[
sa(1 - b'b) = 0 .
\]
Thus,
\[a = a(1 - b'b) \text{ fails to have a right inverse.}
\]

Suppose, conversely, that \( a \) fails to have a right inverse. If one of \( a \) or \( b \), say \( a \), fails to have a right inverse, then, since \( R \) is regular, there is an \( x \) in \( R \) such that
\[
axa = a .
\]
Thus,
\[
(1 - ax)a + 0 \cdot b = 0 \quad \text{and} \quad s = (1 - ax) + 0 .
\]
So, we may assume that there are \( a' \) and \( b' \) in \( \mathcal{R} \) such that
\[
    a' a = b' b = 1
\]
using the regularity of \( \mathcal{R} \) to find a \( y \in \mathcal{R} \) such that \( a y a = a' \), we have
\[
    (1 - a' y) a - (1 - a y) a b' b = (1 - a y) a = 0
\]
and, by assumption, \( s = (1 - a y) \neq 0 \). Hence \( \mathcal{R} \) is dependent. This concludes the proof of Theorem 9.

An immediate consequence of Theorem 9 is:

**Corollary 10.** If \( \mathcal{R} \) is a regular ring in which one-sided inverses are two-sided, then \( \mathcal{R} \) is dependent.

We conclude this section by giving an example of a dependent regular ring which has elements with one-sided inverses that are not two-sided, and hence, by Proposition 2, is not unit-regular.

**Example 11.** A dependent ring with one-sided inverses that are not two-sided.

Let \( \mathcal{R} \) be any regular ring with one-sided inverses that fail to be two-sided (e.g., the ring of all linear transformations of an infinite dimensional vector space over a division ring). Let \( \mathcal{J} \) denote the ring of all sequences \( \langle a_n \rangle \) of elements of \( \mathcal{R} \) all but finitely many of whose terms lie in the center of \( \mathcal{R} \). Since the center of a regular ring is regular [8, p. 132], it follows easily that \( \mathcal{J} \) (with the usual coordinatewise operations) is regular, and has elements with right inverses that are not left inverses.

Suppose \( \langle a_n \rangle, \langle b_n \rangle, \langle a'_n \rangle, \langle b'_n \rangle \) are elements of \( \mathcal{J} \) such that
\[
    \langle a_n a'_n \rangle = \langle b_n b'_n \rangle = \langle 1 \rangle.
\]

Then, there is a positive integer \( N \) such that both \( b_n \) and \( b'_n \) are in the center of \( \mathcal{R} \) for \( n \geq N \). Hence
\[
    (1 - b'_n b_n) = 0 \quad \text{for} \quad n \geq N
\]
so
\[
    \langle a_n \rangle \langle 1 - b'_n \rangle \langle b_n \rangle
\]
cannot have a right inverse. Hence, by Theorem 9, \( \mathcal{J} \) is dependent.

3. Some remarks and problems. (A) The matrix rings over strongly regular rings are precisely the homogeneous \( n \)-regular rings (with identity) described in [6]. Note that each element of such a ring has index of nilpotency no larger than \( n \).
(B) Homomorphic images and complete direct sums of unit-regular rings are unit-regular, so, by Corollary 8, a unit-regular ring can have nilpotent elements of arbitrarily large index; e.g., take \( \mathcal{R} = \mathcal{D}_1 \oplus \mathcal{D}_2 \oplus \cdots \oplus \mathcal{D}_n \oplus \cdots \), for any division ring \( \mathcal{D} \).
(C) There are simple unit-regular rings without any minimal one-sided ideals. For example, if \( V \) is a vector space of countably infinite dimension over a division ring \( \mathcal{D} \), the subring of \( \mathcal{L}(V) \) of all linear transformations representable by matrices of the form
where $A$ ranges over $D_n$, and $n = 1, 2, \ldots$, is such a ring. (I am indebted for this example and other valuable comments to James D. MacKnight, Jr.).

(D) Rings satisfying condition (*) of Proposition (8) have been studied in a variety of contexts, and Proposition 8 can also be deduced from [3, Theorem 4]. Also, as noted by Kaplansky, one need only assume that $aR + bR = R$ implies the existence of a $t \in R$ such that $a + bt$ has a right inverse to conclude that (*) holds. Finally we note that examination of the proof of Proposition 8 yields that if $R$ is any regular ring, and $eR + bR = R$, where $e$ is an idempotent, then there is a $t \in R$ such that $e + bt$ is a unit.

(E) Is there an example of a regular ring (with identity) in which all one-sided inverses are two-sided that is not unit-regular?

(F) Can a regular ring (with identity) be an elementary divisor ring without being unit-regular?

An obvious necessary condition for a ring $R$ to be an elementary divisor ring is that it be (left) dependent. For, if $R$ fails to be dependent, there are $a, b \in R$ such that

$$as + bt = 0$$

implies $s = t = 0$. Hence if

$$\begin{bmatrix} 0 & 0 \\ a & b \end{bmatrix} \begin{bmatrix} s & u \\ t & v \end{bmatrix} = \begin{bmatrix} 0 & u \\ 0 & v \end{bmatrix}$$

is lower triangular, the matrix $\begin{bmatrix} s & u \\ t & v \end{bmatrix}$ cannot be non-singular. So $R$ is not right Hermite, and hence is not an elementary divisor ring.

As noted in [2], the ring of all linear transformations of an infinite dimensional vector space fails to be dependent, so not every regular ring is an elementary divisor ring.

References

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