Two Classes of Rings Generated by Their Units

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In 1953 and 1954, K. Wolfson and D. Zelinsky showed, independently, that every element of the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is a sum of two nonsingular ones, see [16] and [17]. In 1958, Skornyakov [15, p. 167] posed the problem of determining which regular rings are generated by their units. In 1969, while apparently unaware of Skornyakov's book, G. Ehrlich [3] produced a large class of regular rings generated by their units; namely, those rings $R$ with identity in which 2 is a unit and are such that for every $a \in R$ there is a unit $u \in R$ such that $aua = a$. (See also [9] where this author obtained other characterizations of such regular rings.) Finally, in [14], R. Raphael launched a systematic study of rings generated by their units, which he calls $S$-rings.

This note is devoted mainly to generalizing two theorems of Raphael. He shows in [14] that if $R$ is any ring with identity, and $n > 1$ is a positive integer, then every element of the ring $R_n$ of all $n \times n$ matrices with entries from $R$ is a sum of $2n^2$ units. In Section 1 I show, under the same assumptions, that every element of $R_n$ is a sum of three units, and I produce a class of rings $R$ such that not every element of $R_n$ is a sum of two units. A variety of conditions are produced that are either necessary or sufficient for every element of $R_n$ to be a sum of two units if $n > 1$.

Raphael shows also in [14] that if $R$ is a ring with identity such that for every $a \in R$ there is a $y \in R$ such that $aya = a$ and $a^2y = ya^2$, and if 2 is a unit of $R$, then every element of $R$ is a sum of four units. In Section 2, I show that if there is a positive integer $n$ such that

(*) for every $a \in R$, there is an $x \in R$ such that $axa = a$ and $a^n x = xa^n$,

and if $\max[2, (n - 1)]$ is a unit of $R$, then every element of $R$ is a sum of a bounded number of units. Primitive rings $R$ satisfying (*) are rather special and every element of such a ring $R$ with identity is a sum of two units of $R$.

Section 3 is devoted to some additional remarks about $S$-rings, and to posing some problems.
1. Rings of Matrices

Before coming to the main business of this section, I introduce some terminology and make some remarks.

Throughout, $R$ will denote a ring with identity element 1. For any positive integer $n$, $U_n(R)$ will denote the set of elements of $R$ that can be written as a sum of no more than $n$ units of $R$, and we let $U(R) = \bigcup_{n=1}^{\infty} U_n(R)$. As in [14], we call $R$ an $S$-ring if $U(R) = R$; that is if $R$ is generated by its units.

Clearly $U(R)$ is always a subring of $R$, but, if $Z$ is the ring of integers, then $U_n(Z)$ fails to be a subring of $Z$ for any positive integer $n$, even though $Z$ is an $S$-ring. If $R = U_n(R)$ for some positive integer $n$, we call $R$ an $(S, n)$-ring.

As is observed in [14]:

1. If $a$ is a quasiregular element of $R$ (in particular, if $a$ is a nilpotent element of $R$) then $a = (a + 1) - 1$ is a sum of two units.

An element $a \in R$ is called unit-regular if there is a unit $u \in R$ such that $a u a = a$, whence $a u$ is idempotent.

In [3], G. Ehrlich (after observing that no Boolean ring with more than two elements is an $S$-ring) that

2. If $a$ is a unit-regular element of $R$, where $u$ is a unit such that $a u a = a$, and 2 is a unit of $R$, then

$$a = \frac{2a u - 1}{2} u^{-1} + \frac{1}{2} u^{-1}$$

is a sum of two units.

She calls a ring $R$ unit-regular if each of its elements is unit-regular, and she shows that every semisimple Artinian ring and every regular ring with identity and without nonzero nilpotent elements is unit-regular. For other characterizations of these rings, see [9].

By (2) every unit-regular ring in which 2 is a unit is an $(S, 2)$-ring. As noted in the introduction, Wolfson and Zelinsky have shown that the ring of all linear transformations of a vector space over a division ring of characteristic not 2 is an $(S, 2)$-ring. Thus, since one-sided inverses in a unit-regular ring are two-sided, regular $(S, 2)$-rings need not be unit-regular even if 2 is a unit.

If $n$ is a positive integer, we denote, as usual, by $R_n$ the ring of all $n \times n$ matrices with entries from $R$. In [14], Raphael shows that if $n > 1$, then $R_n$ is an $(S, 2n^2)$-ring. In Theorem 3 below it is shown that every such $R_n$ is an $(S, 3)$-ring. I begin by proving a lemma.
Lemma 1. If $R$ is a ring with identity element and $n > 1$, then every diagonal matrix in $R_n$ is a sum of two units of $R_n$.

Proof. Let $D = \text{diag}(a_1, a_2, \ldots, a_n)$ be a diagonal matrix, and let

$$U_1 = \begin{bmatrix}
a_1 & 0 & 0 & \cdots & 0 & 1 \\
1 & a_2 & 0 & \cdots & 0 & 0 \\
0 & 1 & a_3 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & a_{n-1} & 0 \\
0 & 0 & 0 & \cdots & 0 & 1
\end{bmatrix}$$

$$U_2 = \begin{bmatrix}
0 & 0 & 0 & \cdots & 0 & -1 \\
-1 & 0 & 0 & \cdots & 0 & 0 \\
0 & -1 & 0 & \cdots & 0 & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\
0 & 0 & 0 & \cdots & -1 & 0 \\
0 & 0 & 0 & \cdots & 0 & -1
\end{bmatrix}$$

Then $D = U_1 + U_2$, and for $i = 1, 2$, $U_i$ can be reduced to the identity matrix by either elementary row or elementary column transformations, and hence is a unit of $R_n$.

The following lemma is due to I. Kaplansky.

Lemma 2 (Kaplansky). If $R$ is any ring with identity and $n \geq 1$, then every element of $R_n$ is the sum of a diagonal matrix and a unit $U$ of $R_n$.

Proof. Since $a = (a - 1) + 1$ for any $a \in R$, the lemma holds if $n = 1$. We proceed by induction on $n$.

Assume next that the lemma holds in $R_n$ for some fixed $n \geq 1$. If $A' \in R_{n+1}$, write

$$A' = \begin{bmatrix} A & B \\ C & d \end{bmatrix},$$

where $A \in R_n$, $B$ and $C$ are $n$-vectors over $R$, and $d \in R$. By assumption, $A = D + U$, where $D$ is a diagonal matrix and $U$ is a unit in $R_n$. Then if

$$D' = \begin{bmatrix} d & 0 \\ 0 & d - 1 - CU^{-1}B \end{bmatrix} \quad \text{and} \quad U' = \begin{bmatrix} U & B \\ C & 1 + CU^{-1}B \end{bmatrix}$$

$A' = D' + U'$. Moreover if $I$ is the unit matrix of $R_n$, then $P'$ is the identity matrix of $R_{n+1}$ and $P$ and $Q$ are units of $R_{n+1}$. Thus $U'$ is a unit of $R_{n+1}$, so the lemma is proved.
Combining these two lemmas yields the following.

**Theorem 3.** If $R$ is any ring with identity, and $n > 1$, then every element of $R_n$ is the sum of three\(^1\) units of $R_n$.

Next, it will be shown that if $R$ is a polynomial ring in $n > 1$ indeterminates over a field, then $R_n$ is not an $(S, 2)$-ring. Two lemmas and a theorem are needed to establish this. The proof of the first is left as an exercise.

**Lemma 4.** An element $x$ of a ring $R$ is the sum of two units of $R$ if and only if there is a unit $u$ of $R$ such that $(ux + 1)$ is a unit.

**Lemma 5.** Suppose $R$ is a ring with identity $n > 1$, and let

$$X = \begin{bmatrix} x_1 & x_2 & \cdots & x_n \\ 0 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{bmatrix}$$

be an element of $R_n$. Then $X$ is the sum of two units of $R_n$ if and only if there are elements $a_{11}, a_{21}, \ldots, a_{n1}$ of $R$ such that

(a) $1 + \sum_{i=1}^{n} x_i a_{i1}$ is a unit of $R$,

and

(b) $[a_{11}, \ldots, a_{n1}]$ is the first column of a unit matrix $U = [a_{ij}]$ of $R_n$.

**Proof.** Suppose $U = [a_{ij}]$ is a unit of $R_n$. If $I'$ is the unit matrix of $R_n$ and $I$ is the unit matrix of $R_{n-1}$, then

$$XU + I' = \begin{bmatrix} (1 + \sum_{i=1}^{n} x_i a_{i1}) & \sum_{i=1}^{n} x_i a_{i2} & \cdots & \sum_{i=1}^{n} x_i a_{in} \\ 0 & \vdots & & \vdots \\ \vdots & & & \vdots \\ 0 & & & I \end{bmatrix}$$

If

$$P = \begin{bmatrix} 1 & (\sum_{i=1}^{n} x_i a_{i2}) & \cdots & (\sum_{i=1}^{n} x_i a_{in}) \\ 0 & \vdots & & \vdots \\ \vdots & & & I \\ 0 & & & \end{bmatrix}$$

\(^1\) In an earlier version of this paper, before discussion with I. Kaplansky, I had been able only to prove this result with “three” replaced by “four.”
then $P$ is a unit of $R_n$ and $P(XU + I') = \text{diag}(1 + \sum_{i=1}^{n} x_i a_{i1}, 1, \ldots, 1)$. Hence $XU + I'$ is a unit of $R_n$ if and only if (a) and (b) hold. Thus Lemma 5 follows from Lemma 4.

**Theorem 6.** Suppose $R$ is a commutative ring with identity, $n > 1$, and $R$ contains prime ideals $P_1, P_2, \ldots, P_n$, and elements $x_1, x_2, \ldots, x_n$ such that

(i) $x_i \in \bigcap_{j \neq i} P_j$ and $x_i \notin P_i$ for $i = 1, 2, \ldots, n$,

and

(ii) $(P_1 + P_2 + \cdots + P_n) \cap U(R) = \{0\}.

Then

$$X = \begin{bmatrix}
  x_1 & x_2 & \cdots & x_n \\
  0 & 0 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & 0
\end{bmatrix}$$

is not the sum of two units of $R_n$.

**Proof.** Suppose $a_{i1}, a_{21}, \ldots, a_{n1}$ are elements of $R$ satisfying (a) of Lemma 5. Then, by (ii), $\sum_{i=1}^{n} x_i a_{i1} = 0$, so, by (i), $x_i a_{i1} = -\sum_{j \neq i} x_j a_{j1} \in P_i$ for $i = 1, 2, \ldots, n$. But $P_i$ is a prime ideal not containing $x_i$, so $a_{i1} \in P_i$. Hence $a_{11}, a_{21}, \ldots, a_{n1}$ are in $P_1 + P_2 + \cdots + P_n$, which is a proper ideal by (ii). Hence $[a_{i1}, a_{i2}, \ldots, a_{in}]$ cannot be the first column of a unit $[a_{ij}]$ of $R_n$. For, if $[b_{ij}]$ is in $R_n$, and $[b_{ij}] [a_{ij}] = [c_{ij}]$, then $c_{11} \in P_1 + P_2 + \cdots + P_n$. So, by Lemma 5, $X$ is not a sum of two units of $R_n$.

**Corollary 7.** (a) If $R = F[x_1, \ldots, x_n]$ is the ring of polynomials in $n$ indeterminates over a field $F$, and $n > 1$, then $R_n$ is not an $(S, 2)$-ring.

(b) If $R$ is the ring of polynomials in countably many indeterminates $x_1, x_2, \ldots, x_n, \ldots$ over a field $F$, then $R_n$ is not an $(S, 2)$-ring for any positive integer $n$.

**Proof.** Let $P_i = \sum_{i \neq j} x_j R$ and apply Lemma 6.

Next, some definitions are introduced to help us give some condition on the ideal structure of a ring $R$ which imply that $R_3$ is an $(S, 2)$-ring.

The unordered pair of elements $a, b$ of a ring $R$ with identity is called an **Hermite pair** if there is a unit $U$ or $R_2$ and a $d \in R$ such that $\begin{bmatrix} a & b \\ d & c \end{bmatrix} U = \begin{bmatrix} a & b \\ d & c \end{bmatrix}$. Note that if $a, b$ is an Hermite pair, then $aR + bR = dR$ is a principal right ideal. If every pair of elements of $R$ is an Hermite pair, then $R$ is called a **right Hermite ring**. In [11, Theorem 3.5], Kaplansky shows that if $R$ is a right Hermite ring, $n \geq 1$, and $A \in R_n$, there is a unit $U$ of $R_n$ such that $AU$ is lower triangular. For more discussion of commutative Hermite rings, see [6], [7], [10], and [13].
Theorem 8. If \( R \) is a ring with identity and \( a, b \) is a pair of elements of \( R \) such that

(i) \((aR + bR) \cap U_2(R) = \{0\},\)

and

(ii) \( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \) is the sum of two units of \( R_2 \),

then \( a, b \) is an Hermite pair.

Proof. By Lemma 5 and (i) there is a unit \( U = \begin{bmatrix} d & e \\ 0 & 0 \end{bmatrix} \) in \( R_2 \) such that

\( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \begin{bmatrix} d & e \\ 0 & 0 \end{bmatrix} = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \) where \( d = aq + bs \). Then \( V = U^{-1} \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \) is a unit and

\( \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} V = \begin{bmatrix} a & b \\ 0 & 0 \end{bmatrix} \), so \( a, b \) is an Hermite pair.

The following corollary, together with Theorem 11 below shows that if \( R \) is a ring of polynomials in \( m \) indeterminates over a field, and \( n > 1 \), then \( R_n \) is an \((S, 2)\)-ring if and only if \( m = 1 \).

Corollary 9. Suppose \( R \) is a ring such that \( U_2(R) \) is a division ring and \( R_2 \) is an \((S, 2)\)-ring. Then \( R \) is an Hermite ring.

Next, I make use of Theorem 8 to present an example (due to I. Kaplansky) of a Dedekind domain \( R \) such that \( R_2 \) is not an \((S, 2)\)-ring.

Example 10. Let \( R = \{a + \beta \sqrt{-5}: a, \beta \in \mathbb{Z}\} \). It is well known (and easily verified) that the only units of \( R \) are 1 and \(-1\), so

\( U_2(R) = \{0, 1, -1, 2, -2\}. \)

Hence \((3R + (2 + \sqrt{-5})R) \cap U_2(R) = \{0\}\). But \(3R + (2 + \sqrt{-5})R\) is not a principal ideal and so \(3, 2 + \sqrt{-5}\) is not an Hermite pair. (For details, see [2, Chapter 3].)

Finally, I present some sufficient conditions on a ring \( R \) in order that \( R_n \) be an \((S, 2)\)-ring if \( n > 1 \).

Two elements \( a, b \) of a ring \( R \) with identity are said to be equivalent if there are units \( p \) and \( q \) of \( R \) such that \( paq = b \). A ring \( R \) is called an elementary divisor ring if, for every positive integer \( n \), every \( A \in R_n \) is equivalent to a diagonal matrix. Every elementary divisor ring is both a left and a right Hermite ring, and an example is given in [7, Example 4.11] of a commutative Hermite ring that is not an elementary divisor ring.

I make use of the fact that if \( A \in R_n \) can be transformed into \( B \in R_n \) by means of elementary row and column transformations, then \( A \) and \( B \) are equivalent.

An immediate consequence of Lemma 1 and the definition of elementary divisor ring follows.

Theorem 11. If \( R \) is an elementary divisor ring and \( n > 1 \), then every element of \( R_n \) is a sum of two units.
THEOREM 12. If \( R \) is a ring with identity such that \( U_n(R) = R \), and \( n > 1 \), then every element of \( R_n \) is a sum of two units.

Proof. Suppose \( A = [a_{ij}] \in R_n \). Two cases are considered.

Case 1. For \( i = 1, 2, ..., n \), there are units \( u_i \) and \( v_i \) of \( R \) such that \( a_{ii} = u_i + v_i \). Let

\[
U = \begin{bmatrix}
  u_1 & 0 & \cdots & 0 \\
  a_{21} & u_2 & \cdots & 0 \\
  \vdots & \vdots & \ddots & \vdots \\
  a_{n1} & a_{n2} & \cdots & u_n \\
\end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix}
  v_1 & a_{12} & \cdots & a_{1n} \\
  0 & v_2 & \cdots & a_{2n} \\
  \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & \cdots & v_n \\
\end{bmatrix}
\]

Each of \( U \) and \( V \) is clearly equivalent to the identity matrix and hence is a unit of \( R \). Thus \( A = U + V \) is the sum of two units of \( R \).

Case 2. Some \( a_{ii} \) is a unit. By interchanging rows and columns we see that \( A \) is equivalent to a matrix whose upper left element is a unit. After multiplying by its inverse, we may use it to sweep out the other nonzero elements of the first row and first column. Thus \( A \) is equivalent to a matrix of the form

\[
A' = \begin{bmatrix}
  1 & 0 & \cdots & 0 \\
  0 & \ddots & & \vdots \\
  \vdots & & \ddots & \vdots \\
  0 & 0 & \cdots & B \\
\end{bmatrix}
\]

where \( B = [b_{ij}] \in R_{n-1} \).

Continuing this process, we may conclude that either \( A \) is equivalent to the identity of \( R_n \) or \( A \) is equivalent to a matrix of the form

\[
A'' = \begin{bmatrix}
  I_r & 0 \\
  0 & B \\
\end{bmatrix}
\]

where \( I_r \) is the identity matrix of \( R_r \), \( B = [b_{ij}] \in R_{n-r} \), \( 1 \leq r < n \), and there are units \( u_i \) and \( v_i \) such that \( b_{ii} = u_i + v_i \) for \( i = 1, 2, ..., n - r \).

If \( A \) is equivalent to the identity matrix or if \( n = 2 \) then \( A \) is the sum of two units by Lemma 1.

If \( n > 2 \) and \( r = 1 \), let

\[
U = \begin{bmatrix}
  1 & 1 & 0 & \cdots & 0 \\
  1 & 0 & b_{13} & \cdots & b_{1n} \\
  0 & 0 & u_2 & \cdots & b_{2n} \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & 0 & 0 & \cdots & u_{n+1} \\
\end{bmatrix} \quad \text{and} \quad V = \begin{bmatrix}
  0 & -1 & 0 & \cdots & 0 \\
  -1 & b_{11} & 0 & \cdots & 0 \\
  0 & b_{21} & v_2 & \cdots & 0 \\
  \vdots & \vdots & \vdots & \ddots & \vdots \\
  0 & b_{n-1,1} & 0 & \cdots & v_{n-1} \\
\end{bmatrix}
\]

Then \( A'' = U + V \). Beginning with the \( n \)th row of \( U \), we may use \( u_{n-1} \) to
sweep out the nonzero elements above it. Repeating this process successively with \(u_{n-2}, \ldots, u_2\), we see that \(U\) is equivalent to
\[
U' = \begin{bmatrix}
1 & 1 & 0 & \cdots & 0 \\
1 & 0 & 0 & \cdots & 0 \\
0 & 0 & u_2 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & u_{n-1}
\end{bmatrix}
\]
and hence is a unit. Similarly \(V\) is a unit, so \(A''\) is the sum of two units of \(R_n\) in this case.

If \(r \geq 2\) and \(r > 2\), then by Lemma 1, there are units \(P_1\) and \(Q_1\) of \(R_r\) such that \(I_r = P_1 + Q_1\), and by Case 1, there are units \(P_2\) and \(Q_2\) of \(R_{n-r}\) such that \(B = P_2 + Q_2\). Then \(\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\) and \(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}\) are units of \(R_n\) and \(A''\) is their sum.

This completes the proof of Theorem 12.

Since by (1), every quasi-regular element of a ring with identity is a sum of two units, we immediately have the following corollary.

**Corollary 13.** If \(R\) is a ring such that every element not in the Jacobson radical of \(R\) is a unit, in particular if \(R\) is a quasilocal (commutative) ring, and \(n > 1\), then \(R_n\) is an \((S, 2)\)-ring.

This corollary shows that finitely generated ideals of an \((S, 2)\)-ring need not be principal. For example, if \(F\) is a field, \(n > 1\), \(S = F[x_1, x_2, \ldots, x_n]\) and \(R\) is the localization of \(S\) at the maximal ideal consisting of all \(f(x_1, x_2, \ldots, x_n) \in S\) such that \(f(0, 0, \ldots, 0) = 0\), then \(R\) is quasilocal. But \(x_1R + x_2R + \cdots + x_nR\) cannot be generated by fewer than \(n\) elements of \(R\).

Another immediate consequence of Theorem 12 follows.

**Corollary 14.** If \(R_n\) is an \((S, 2)\)-ring and \(n \geq 1\), then \(R_{nk}\) is an \((S, 2)\)-ring for any positive integer \(k\).

I have been unable to assemble this mélange of necessary conditions and sufficient conditions on a ring \(R\) in order that \(R_n\) be an \((S, 2)\)-ring into a condition that is simultaneously necessary and sufficient. For further discussion, see Problem C in Section 3.

2. Some Regular Rings in Which Every Element Is a Sum of a Bounded Number of Units

In [14], Raphael shows that if for every \(a \in R\), there is a \(\gamma \in R\) such that \(a\gamma = a\) and \(a^2\gamma = \gamma a^2\), then \(R\) is a \((S, 4)\)-ring. We generalize this result in Theorem 17.
The next observation is part of the proof of Proposition 8(a) in [14].

**Lemma 15** (Raphael). If \( b \) is an element of \( R \) for which there is an \( x \in R \) such that \( bxb = b \) and \( bx = xb \), then \( b \) is unit-regular.

**Proof.** Let \( y = xb \). Then \( byb = b \) and \( by = bxb = bx = xbxb = yb \). Let \( z = 1 - yb + y \). It is easily verified that \( bzb = b \) and \( z^{-1} = 1 - yb + b \). Hence \( b \) is unit-regular.

**Lemma 16.** Suppose \( R \) is a ring in which \( 2 \) is a unit. If \( a \) is an element of \( R \) for which there is an integer \( n > 1 \) and an \( x \in R \) such that \( axa = a \) and \( a^{n}x = xa^{n} \), then \( a^{n-1} \) is a sum of four units of \( R \).

**Proof.** Clearly \( a^{n-1} = (a^{n-1} - a^{n}x) + a^{n}x \). It is easily verified that if \( z = (a^{n-1} - a^{n}x) \), then \( z^{2} = 0 \), so, by (1), \( z \in U_{2}(R) \).

Let \( w = a^{n}x \). Then since \( a^{n} \) commutes with \( x \), \( w^{n-1} \). Moreover, since \( a^{n}x = xa^{n} \) and \( axa = a \), it is clear that \( a^{n+1}x = a^{n}x = xa^{n+1} \). Hence

\[
wx^{n-1}w = a^{n}x^{n+1} = a^{n-1}(a^{n+1}x)^{n} = a^{n-1}x^{n} = a^{n-2}x^{n-1} = \cdots = a^{n}x = w.
\]

Thus by Lemma 15 and (2), \( w \in U_{2}(R) \). It follows that \( a^{n-1} \in U_{4}(R) \).

Combining these two lemmas yields the theorem.

**Theorem 17.** Suppose \( R \) is a ring for which there is a positive integer \( n \) such that

(a) \( \max[2,(n-1)!] \) is a unit of \( R \) and

(b) for every \( a \in R \) there is an \( x \in R \) such that \( axa = a \) and \( a^{n}x = xa^{n} \).

Then every element of \( R \) is the sum of a bounded number of units.

**Proof.** By Lemma 15, \( R = U_{2}(R) \) if \( n = 1 \). If \( n > 1 \), then Lemma 16 and the well known identity

\[
(n - 1)! \ y = \sum_{r=0}^{n-2} (-1)^{n-2-r} \binom{n-2}{r} (y + r)^{n-1} \frac{(n - 2)[(n - 1)!]}{2}
\]

completes the proof of the theorem, see [8, pp. 325–26].

The proof of Theorem 17 enables us, for any ring \( R \) satisfying its hypotheses, to calculate an upper bound \( N \), depending on \( n \), such that \( R = U_{N}(R) \). In general, this upper bound is much too conservative, for many identities like (3) exist for various values of \( n \) which enable us to express \( x \) as a sum of fewer \( (n - 1) \)st powers assuming that (a) holds. (See [4] and the bibliography concerned with the easier Waring problem.)
The next theorem, which is due essentially to Kaplansky, indicates that much more can be said about primitive rings satisfying the hypotheses of Theorem 17.

**Theorem 18.** Suppose $R$ is a primitive ring with identity for which there is a positive integer $n$ such that for every $a \in R$, there is an $x \in R$ satisfying $axa = a$ and $a^{n+1}x = xa^n$. Then either $R$ is a division ring $D$ or the algebra of $N \times N$ matrices over $D$ for some positive integer $N$. In either case, every element of $R$ is a sum of two units unless $R$ is a two-element field.

**Proof.** As noted in the proof of Lemma 16, the hypotheses imply that $a^{n+1}x = xa^n$, so $R$ is a primitive regular ring with identity whose nilpotent elements have index no larger than $n$. Hence the conclusion follows from [12, Theorem 2.3] and Theorem 11.

3. Remarks and Problems

(A) In [14], Raphael asks if $eRe$ is an $S$-ring whenever $e$ is an idempotent of an $S$-ring $R$. I answer this in the negative by means of the following example.

Let $T$ be any ring with identity that is not an $S$-ring, let $R = T_0$, and let $e = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$. If $a = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} \in R$, then $eae = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$, so $eRe$ is isomorphic to the non-$S$-ring $T$, even though $R$ is an $S$-ring by Theorem 2. Indeed, if we choose $T$ to be the ring $F[x]$ of all polynomials over a field, $eRe$ will fail to be an $S$-ring even though $R$ is an $(S, 2)$-ring by Theorem 11.

(B) By making use of an extension of the Cartan–Brauer–Hua theorem due to S. Amitsur [1], C. Faith showed [5, Theorem 5] that if $R$ is a simple algebra over a field with more than two elements, and $R$ has an idempotent $e \neq 0$ or 1, then $R$ is generated by its quasi-regular elements. Hence if $R$ has an identity, it is an $S$-ring. The techniques used do not reveal if $R = U_n(R)$ for some positive integer $n$, and it seems natural to ask if this is the case.

(C) The fact that $U_n(R)$ does not seem to be closed under any reasonable algebraic operation makes it difficult to imagine the existence of a characterization of rings $R$ such that for $n > 1$, $R_n$ (or even $R_3$) is an $(S, 2)$-ring, but the problem seems worthy of more examination—at least in special cases. In particular, if $R$ is a (right) Hermite ring, is $R_3$ an $(S, 2)$-ring? What if, in addition, $U_3(R)$ is a division ring and/or if $R$ is commutative? The only known example of a commutative Hermite ring that is not an elementary divisor ring is the ring of all real-valued continuous functions on a certain topological space, see [7, Example 4.11]. As Raphael notes in [14],
the ring of all real-valued continuous functions on any topological space is an $(S, 2)$-ring, so, by Theorem 12, this example does not provide a negative answer to this latter question.

Corollary 14 shows that if $R_2$ is an $(S, 2)$-ring, so is $R_{2k}$ for any $k \geq 1$. It is true that if $R_2$ is an $(S, 2)$-ring, then so is $R_n$ for any $n \geq 2$?

(D) Suppose $R$ is a ring with identity in which (*) holds. By Theorem 18, every such ring is a subdirect sum of total matrix algebras over division rings, and hence is a subdirect sum of unit-regular rings.

In every regular ring with identity that is a subdirect sum of unit-regular rings necessarily unit-regular?

An affirmative answer to this question would show that any ring $R$ satisfying (*) and in which 2 is a unit is an $(S, 2)$-ring. Call a ring $R$ with identity Von Neumann finite if $a, b \in R$ and $ab = 1$ implies $ba = 1$, and note that a ring with identity that is a subdirect sum of Von Neumann finite rings is also Von Neumann finite. Thus, a negative answer to this question would yield an example of a regular Von Neumann finite ring that is not unit-regular. This would answer question (E) of [9] in the negative.

(E) In [14], Raphael shows that if it is true that every regular ring $R$ with identity in which 2 is a unit is an $S$-ring, then there is a positive integer $n$ such that $R = U_n(R)$, where $n$ does not depend on $R$. I conclude by posing the additional questions: Is there a regular $S$-ring $R$ such that $R \neq U_2(R)$? What if 2 is a unit in $R$?

Note added in proof. Recently G. Ehrlich has shown that every Von Neumann regular ring is unit-regular, thereby answering the question in (D) in the affirmative.

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References


