A SIMPLE CHARACTERIZATION OF COMMUTATIVE RINGS
WITHOUT MAXIMAL IDEALS

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In a course in abstract algebra in which the instructor presents a proof that each
deal in a ring with identity is contained in a maximal ideal, it is customary to give
an example of a ring without maximal ideals. The usual example is a zero-ring whose
additive group has no maximal subgroups (e.g., the additive group of (dyadic)
rational numbers; actually any divisible group will do; see [1, p. 67]). This may
leave the impression that all such rings are artificial or at least that they abound
with divisors of 0.

Below, I give a simple characterization of commutative rings without maximal
ideals and a class of examples of such rings, including some without proper divisors
of 0. To back up the contention that this can be presented in such a course in abstract
algebra, I outline proofs of some known theorems including a few properties of
radical rings in the sense of Jacobson.

The Hausdorff maximal principle states that every partially ordered set contains
a maximal chain (i.e., a maximal linearly ordered subset). It is equivalent to the
axiom of choice [4, Chapter XI].

Since the union of a maximal chain of proper ideals in a ring with identity is a
maximal ideal, and since the union of a maximal chain of linearly independent
subsets of a vector space is a maximal linearly independent set, we have:

(1) Every ideal in a ring with identity is contained in a maximal ideal.
(2) Every non-zero vector space has a basis.

As usual we denote the ring of integers by \( Z \), and for any prime \( p \in Z \), we denote
by \( Z_p \) the ring of integers modulo \( p \), and by \( Z'_p \) the zero-ring whose additive group is
the same as that of \( Z_p \).

It is not difficult to prove that a commutative ring \( R \) has no nonzero proper
ideals if and only if either \( R \) is a field or \( R \) is isomorphic to \( Z'_p \) for some prime \( p \).
See [5, p. 133]. Hence:

(3) An ideal \( M \) of a commutative ring \( R \) is maximal if and only if \( R/M \) is either
a field or is isomorphic to \( Z'_p \) for some prime \( p \).

For any commutative ring \( R \), let \( J(R) \) denote the intersection of all the ideals \( M \)
of $R$, such that $R/M$ is a field. If $R$ has no such ideals, let $J(R) = R$. In the latter case we call $R$ a radical ring. The knowledgeable reader will recognize $J(R)$ as the Jacobson radical of $R$. See [2, Chapter 1].

Of the many known properties of radical rings, we need only the following two, the first of which follows immediately.

(4) A homomorphic image of a (commutative) radical ring is a radical ring.
(5) $J(R)$ is a radical ring.

Proof. If $J(R)$ is not a radical ring, then there is a homomorphism $\phi$ of $J(R)$ onto a field $F$ with identity element 1. Choose $e \in J(R)$ such that $\phi(e) = 1$, and define $\phi': R \to F$ by letting $\phi'(a) = \phi(ae)$ for each $a \in R$. If $a, b \in R$, then

$$
\phi'(a + b) = \phi((a + b)e) = \phi(ae + be) = \phi(ae) + \phi(be) = \phi'(a) + \phi'(b),
$$

and $\phi'(ab) = \phi(abe) = \phi(a)e\phi(b) = \phi(a)\phi'(b).

Therefore $\phi'$ is a homomorphism of $R$ onto $F$, and hence its kernel contains $J(R)$. But $e \in J(R)$ and $\phi'(e) = 1$. This contradiction shows that $J(R)$ is a radical ring.

It follows easily from (1), (3), and (4) that no ring with identity is a radical ring and that every zero-ring is a radical ring.

**Theorem.** A commutative ring $R$ has no maximal ideals if and only if

(a) $R$ is a radical ring.
(b) $R^2 + pR = R$ for every prime $p \in \mathbb{Z}$.

Proof. Suppose first that (a) and (b) hold. Since $R$ is a radical ring, no homomorphic image of $R$ can be a field, so, by (3) it suffices to show that for any prime $p \in \mathbb{Z}$, the zero-ring $Z_p$ is not a homomorphic image of $R$. Suppose, on the contrary, that there is a homomorphism $\phi$ of $R$ onto $Z_p$ with kernel $M$. If

$$
c = \sum_{i=1}^n a_i b_i \in R^2,
$$

then $\phi(c) = \sum_{i=1}^n \phi(a_i)\phi(b_i) = 0,$

so $R^2 \subseteq M$. Moreover, $\phi(pa) = p\phi(a) = 0$, so $pR \subseteq M$. Hence $R^2 + pR \subseteq M \neq R$, so (b) fails. The contradiction shows that $R$ has no maximal ideals.

Suppose next that $R$ has no maximal ideals. By (3) and the definition of $J(R)$, $R$ is a radical ring. Suppose (b) fails for some prime $p$, let $I = R^2 + pR$, and let $\phi$ be the natural homomorphism of $R$ onto $R/I$. If $a, b \in R$, then $0 = \phi(ab) = \phi(a)\phi(b)$, so $R/I$ is a zero-ring, and since $0 = \phi(pa) = p\phi(a) = 0$, $R/I$ has characteristic $p$ and hence is a vector space over $Z_p$. By (2), since $I \subseteq R$, $R/I$ has a basis $\{x_\alpha\} \subseteq R$ and each $x \in R/I$ may be written uniquely as $x = \sum_{\alpha \in \Gamma} a_\alpha x_\alpha$ with $a_\alpha \in Z_p$ and $a_\alpha = 0$ for all but finitely many $\alpha \in \Gamma$. For any fixed $\alpha_0 \in \Gamma$, the mapping $\psi_0$ such that $x\psi_0 = a_\alpha x_\alpha$ is a homomorphism of $R/I$ onto $Z_p$. Then $\phi \circ \psi_0$ is a homomorphism of $R$ onto $Z_p$. By (3), the kernel of $\phi \circ \psi_0$ is a maximal ideal, contrary to assumption. Hence (a) and (b) hold.
Recall that an abelian group $G$ is divisible if $nG = G$ for every $n \in \mathbb{Z}$ and note that $G$ is divisible if and only if $pG = G$ for every prime $p \in \mathbb{Z}$. It follows from the theorem that a zero-ring whose additive group is divisible has no maximal ideals.

**Corollary.** Let $S$ be a commutative ring with identity that has a unique maximal ideal $R$. If $R^2 + pR = R$ for every prime $p \in \mathbb{Z}$, then $R$ has no maximal ideals. In particular, if the additive group of $S$ is divisible, then $R$ has no maximal ideals.

I conclude with some explicit examples:

**Examples.** (i) For a field $F$, let $F[x]$ denote the ring of polynomials in an indeterminate $x$ with coefficients in $F$, and let $F(x)$ denote the field of quotients of $F[x]$. Let

$$S(F) = \left\{ h(x) = \frac{f(x)}{g(x)} \in F(x) : f(x), g(x) \in F[x] \text{ and } g(0) \neq 0 \right\}.$$  

It is easy to verify that $S(F)$ is an integral domain whose unique maximal ideal is $R(F) = xS(F)$. If $F$ has characteristic zero, then, by the corollary, $R(F)$ has no maximal ideals. If $F$ has prime characteristic, then, since $[R(F)]^2 = x^2 R(F)$, the ring $R(F)$ does have maximal ideals.

(ii) Let $G$ denote the additive semigroup of non-negative dyadic rational numbers, and let $U(F)$ denote the semigroup algebra over $G$ with coefficients in a field $F$. We may regard each element of $U(F)$ as a polynomial in $x^{(n)}$ for some positive integer $n$. Let $T(F)$ denote those elements of the quotient field of $U(F)$ whose denominators fail to vanish at 0. It is not difficult to verify that $R^*(F) = \{ h \in T(F) : h(0) = 0 \}$ is the unique maximal ideal of $T(F)$ and that $[R^*(F)]^2 = R^*(F)$. By the corollary, $R^*(F)$ has no maximal ideals (and no proper divisors of 0).

(iii) Let $F_1$ be a field of characteristic 0, let $F_2$ be a field of prime characteristic $p$, and let $R$ be the direct sum of the ring $R(F_1)$ described in (i) and the ring $R^*(F_2)$ described in (ii). Since each of these latter two rings is a radical ring, so is $R$. For, otherwise, there would be a homomorphism $\phi$ of $R$ onto a field $F$. Then $\phi[R(F_1)]$ and $\phi[R^*(F_2)]$ are ideals of $F$ whose (direct) sum is $F$, and hence one of them is all of $F$, contrary to the fact that $R(F_1)$ and $R^*(F_2)$ are radical rings. Also, while $R^2 \neq R$ and $pR \neq R$, it is true that $R^2 + pR = R$, so $R$ has no maximal ideals.

One can create more rings satisfying the hypothesis of the corollary by starting with any commutative ring $S$ with identity and divisible additive group, taking its localization $S_M$ at a maximal ideal $M$, and letting $R = MS_M$. See [1, Chapter 2].

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**References**


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