SOME SUFFICIENT CONDITIONS FOR
THE JACOBSON RADICAL OF A COMMUTATIVE
RING WITH IDENTITY TO ContAIN A PRIME IDEAL

BY

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1. Introduction

Throughout, the word «ring» will abbreviate the phrase «commutative ring with
identity element 1» unless the contrary is stated explicitly. An ideal I of a ring R is
called pseudoprime if \( ab = 0 \) implies a or b is in I. This term was introduced by C.
Kohls and L. Gillman who observed that if I contains a prime ideal, then I is
pseudoprime, but, in general, the converse need not hold. In
[9 p. 233], M. Larsen, W. Lewis, and R. Shores ask if whenever
the Jacobson radical J(R) of an arithmetical ring is pseudoprime,
it follows that J(R) contains a prime ideal?

In Section 2, I answer this question affirmatively. Indeed, if R
is arithmetical and J(R) is pseudoprime, then the set N(R) of
nilpotent elements of R is a prime ideal (Corollary 9). Along the
way, necessary and sufficient conditions for J(R) to contain a prime
ideal are obtained.

In Section 3, I show that a class of rings introduced by
A. Bouvier [1] are characterized by the property that every minimal
prime ideal of R is contained in J(R). The remainder of the section is
dedicated to rings with pseudoprime Jacobson radical that satisfy

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a variety of chain conditions. In particular, it is shown that if \( R \) is a Noetherian multiplication ring with pseudoprime Jacobson radical \( J(R) \), then \( J(R) \) contains a unique minimal prime ideal (Theorem 20), but there is a Noetherian semiprime ring \( R \) such that \( J(R) \) is pseudoprime and fails to contain a prime ideal (Example 21).

2. **The ideal \( mI \) and pseudoprime ideals**

As in [5], if \( I \) is an ideal of a ring \( R \), let

\[
mI = \bigcup \{ A(I - i) : i \in I \}
\]

where \( A(a) = \{ x \in R : ax = 0 \} \). In [5], the following assertions are proved.

1. **Lemma (Jenkins-McKnight)** If \( I \) and \( K \) are ideals of a ring \( R \) and \( I \subseteq K \), then

   (a) \( mI \) is an ideal of \( R \) contained in \( I \)

   (b) \( mI = \{ axR : I + A(a) = R \} \)

   (c) \( mI \subseteq mK \)

   (d) \( m(I + J(R)) = mI \).

Recall that the Jacobson radical \( J(R) \) of a commutative ring \( R \) with identity is the intersection of all the maximal ideals of \( R \), and that \( axJ(R) \) if and only if \( (1 - ax) \) is a unit for every \( x \in R \) [11, Section 30].

Let \( U(R) \) denote the set of units of a ring \( R \), let \( M(R) \) denote the set of maximal ideals of \( R \), and let \( S(R) = \Sigma \{ mI : mI \text{ is a proper ideal of } R \} = \Sigma \{ A(1 - i) : i \in R \} \backslash U(R) \) is the smallest ideal containing \( A(1 - i) \) for every non unit \( i \in R \).

The next lemma indicates the importance of the ideals \( mI \) in the study of rings with pseudoprime Jacobson radical.

2. **Lemma.** The Jacobson radical \( J(R) \) of a ring \( R \) is pseudoprime if and only if \( S(R) \subset J(R) \).

**Proof.** To prove the lemma, it suffices to show that \( J(R) \) is pseudoprime if and only if \( mI \subset J(R) \) for every proper ideal \( I \) of \( R \).
If $J(R)$ is pseudoprime, I is a proper ideal of $J(R)$, and $aemI$, there is an $i \in I$ such that $a(1 - i) = 0$. But $(1 - i) \notin J(R)$, so $aemI$. 

Suppose, conversely, that $mI \subset J(R)$ for every proper ideal I of R, $ab = 0$, and $b \notin J(R)$. Then there is an $x \in R$ such that $1 - bx$ is not a unit. Thus $a(1-bx) = a$, so $aem(1-bx)I \subset J(R)$. 

Suppose I is a proper ideal of a ring R (which need not have an identity element). A proper prime ideal of R that fails to contain properly any other prime ideal of R is said to be a minimal prime ideal of R. Let $\mathcal{P}(R)$ denote the set of minimal prime ideals of R. It is well known that $\bigcap \{P: P \in \mathcal{P}(R)\}$ is the set N(R) of nilpotent elements of R [11, p. 100], and that a prime ideal P is minimal if for every $a \in P$, there is a $b \notin P$ such that $abc \in N(R)$ [5, lemma 3.1]. 

If $N(R) = \{0\}$, then R is called a semiprime ring. 

For any ideal I of R, the radical $\sqrt{I}$ of I is the intersection of all the prime ideals of R containing I. Equivalently, $\sqrt{I} = \{a: a^n \in I\}$ for some positive integer n. The next proposition describes $\sqrt{mI}$ as an intersection of minimal prime ideals of R. 

3. Proposition. Suppose I is a proper ideal of R and P is a minimal prime ideal of R 

(a) $mI \subset P$ if and only if $I + P \neq R$ 

(b) $\sqrt{mI} = \bigcap \{P \in \mathcal{P}(R): I + P \neq R\}$ 

(c) If M is a maximal ideal of R, then 
\[ \sqrt{mM} = \bigcap \{P \in \mathcal{P}(R): P \subset M\} \] 

(d) If R is semiprime, then $\sqrt{mI} = mI$. 

Proof of (a). If $I + P = R$, there is an $i \in I$ and a $p \in P$ such that $i + p = 1$. Since $P \in \mathcal{P}(R)$, there is a $q \in P$ such that $q(1 - i) = q \in N(R)$. Hence there is a positive integer n such that $q^n(1 - i)^n = 0$. By the binomial theorem $(1 - i)^n = (1 - i')$ for some $i' \in I$, so $q^nemI \in P$. We have shown that $I + P = R$ implies $mI \neq P$.

If, conversely, there is an $a \in mI \in P$, then $a(1 - i) = 0$ for some $i \in I$. Since $a \notin P$, $1 - i \in P$ and $I + P = R$. This completes the proof of (a).
To get (b) from (a), it suffices to show that \( \sqrt{mI} \) is the
intersection of all the minimal prime ideals containing it. It follows
from [7, Theorem 10], that \( mI \) is the intersection of all the prime
ideals of \( R \) such that \( P/mI \) is a minimal prime ideal of \( R/mI \).
Suppose \( a \) is an element of such a prime ideal \( P \). Then there is a \( b \notin P \) and a
positive integer \( n \) such that \( (ab)^n \in mI \). Hence \( ab^n(1-i) = 0 \) for some \( i \in I \).
Suppose \( b^n(1-i) \in P \). Now \( b \notin P \), so \( (1-i)b \in P \) since \( P \) is a prime ideal.
Thus \( I + P = R \), and by (a), \( mI \notin P \). This contradiction shows that
\( ab^n(1-i) = 0 \) and \( b^n(1-i) \notin P \). Hence \( P \notin \mathcal{P}(R) \) and (b) holds.

Clearly (c) follows from (b).

If \( a \notin mI \), then \( a \in mI \) for some positive integer \( n \). So there is
an \( i \in I \) such that \( a^n(1-i) = 0 = [a(1-i)]^n \). Since \( R \) is semiprime,
\( a(1-i) = 0 \) and \( a \in mI \). Thus (d) holds.

For any ring \( R \), let \( G(R) \) denote the multiplicative semigroup
generated by \( \{1-i: i \in \mathcal{U}(R) \} \) and let \( T(R) = \{aeR: ae = 0 \text{ for some } aeG(R) \} \). Note that \( T(R) \) is an ideal of \( R \) which is proper if and only if \( 0 \notin G(R) \). Also, \( S(R) \subseteq T(R) \). For, if \( aeS(R) \), then there
is a finite set \( \{M_1,...,M_n\} \) of maximal ideals, and elements \( m_i \in M_i \)
for \( i = 1,...,n \) such that \( ae\Sigma_{i=1}^n A(1-m_i) \). Then \( a\Pi_{i=1}^n(1-m_i) = 0 \),
so \( aeT(R) \).

4. Proposition. The following properties of a minimal prime \( P \) of
a ring \( R \) are equivalent

(a) \( P \subseteq J(R) \).

(b) \( P \supseteq S(R) \).

(c) \( P \supseteq T(R) \).

Proof. If \( P \subseteq J(R) \) and \( M \) is a maximal ideal of \( R \), then \( P \subseteq M \).
Hence by Proposition 3, \( mM \subseteq P \), so \( S(R) = \Sigma \{mM: M \in \mathcal{M}(R) \} \subseteq P \).
Thus (a) implies (b).

Suppose next that there is an \( aeT(R)/P \). Then there is an
\( aeG(R) \) such that \( ae = 0 \in P \). Since \( a \notin P \), we have \( aeP \).
Since \( aeG(R) \), there is finite set \( \{M_1,...,M_n\} \) of maximal ideals of \( R \) and elements \( m_i \in M_i \) for \( i = 1,...,n \) such that \( x = (1-m_1)...(1-m_n)eP \).
Hence \( (1-m_i)eP \) for some \( i \), so \( P + M_i = R \). By Proposition 3,
\( mM \notin P \) and therefore \( P \notin S(R) \). Thus we have shown that (b)
implies (c).
If \( P \supset T(R) \), then \( P \supset S(R) \supset mM \) for every maximal ideal \( M \) of \( R \). So, by Proposition 3, \( P \subset J(R) \). Thus (c) implies (a) and the proof of Proposition 4 is complete.

Since every proper ideal of \( R \) is contained in a prime ideal, the following corollary follows immediately from Proposition 4 and the remarks preceding it. It may also be derived easily from [2, Proposition 3.3].

5. Corollary. The Jacobson radical of a ring \( R \) contains a prime ideal if for every positive integer \( n \), whenever \( m_1, \ldots, m_n \) is a finite set of non units of \( R \), it follows that 
\[
\prod_{i=1}^{n} (1 - m_i) \neq 0.
\]

Another easy consequence of Proposition 4 follows.

6. Corollary. If \( R \) is a ring with pseudoprime Jacobson radical \( J(R) \), and \( P \) is a minimal prime ideal of \( R \) such that \( P = J(R) \), then \( P = J(R) \).

Proof. Since \( J(R) \) is pseudoprime and \( P \supset J(R) \), then \( P \supset J(R) \supset S(R) \). Hence by Proposition 4, \( P \subset J(R) \), so \( P = J(R) \).

The next theorem and its corollaries solves the problem posed by M. Larsen, W. Lewis, and R. Shores in [9, p. 233]. Recall that if \( I_1 \) and \( I_2 \) are proper ideals of a ring \( R \) and \( I_1 + I_2 = R \), then \( I_1 \) and \( I_2 \) are said to be co-maximal.

7. Theorem. Suppose \( R \) is a ring with pseudoprime Jacobson radical.

(a) If \( S(R) \) contains a prime ideal \( P \), then \( P = S(R) \) is the unique minimal prime ideal of \( R \) contained in \( J(R) \).

(b) If \( \sqrt{mP} \) is a prime ideal, then \( P = N(R) \).

Proof. The prime ideal \( P \) contains a minimal prime ideal \( P_0 \), and by Lemma 2, \( P_0 \subset P \subset S(R) \subset J(R) \). By Proposition 4, \( S(R) \subset P_0 \), so \( P_0 = P = S(R) \). Using Proposition 4 again yields that \( S(R) \) is the unique minimal prime ideal contained in \( J(R) \), and (a) holds.

If \( \sqrt{mP} = Q \) is a prime ideal, then by Proposition 4, \( Q \in \mathcal{P}(R) \). But \( \sqrt{mP} \subset P \), so \( \sqrt{mP} = P \subset J(R) \). By Lemma 1, \( mP = m\sqrt{mP} = \{0\} \). Hence \( P = \sqrt{\{0\}} = N(R) \), and (b) holds.
12. Lemma. If $R$ is a semiprime ring (not necessarily with an identity element) and $\mathcal{P}(R)$ is finite, then $R$ satisfies the annihilator condition.

Proof. By Lemma 11 (a,b,c), if $S \subseteq \mathcal{P}(R)$, there is an $a \in R$ such that $h(a) = S$. Hence if $x, y \in R$, there is a $z \in R$ such that $h(z) = h(x) \cap h(y)$. By [4, Lemma 3.1], since $R$ is semiprime, $A(z) = A(x) \cap A(y)$ and $R$ an a.c.-ring.

13. Proposition. The following properties of an a.c.-ring $R$ such that $\mathcal{P}(R)$ is compact are equivalent

(a) $J(R)$ contains a prime ideal of $R$.

(b) $S(R)$ is not a regular ideal.

Proof. If (a) holds, then $J(R)$ contains a $P \in \mathcal{P}(R)$, by Proposition 4, $S(R) \subseteq P$. But no element of a minimal prime ideal is regular, so (b) holds.

If (b) holds, then by Lemma 11 (c), $S(R)$ is contained in some $P \in \mathcal{P}(R)$. So by Proposition 4, (b) holds.

The following corollary is an immediate consequence of Lemma 12 and Proposition 13.

14. Corollary. If $R$ is a semiprime ring such that $\mathcal{P}(R)$ is finite, then $J(R)$ contains a prime ideal if and only if $S(R)$ is not a regular ideal.

15. Remarks. (a) The hypothesis of Corollary 14 is satisfied if $R$ is a semiprime ring that satisfies the ascending chain condition on annihilator ideals [7, Theorem 88], or if $R$ has few zero divisors in the sense of [10, p. 152].

(b) Since $N(R) \subseteq J(R)$ and $N(R) \subseteq P$ for every $P \in \mathcal{P}(R)$, it follows easily that $J(R)$ is pseudoprime (resp. $J(R)$ contains a prime ideal of $R$) if and only if $J(R/N(R))$ is pseudoprime (resp. $J(R/N(R))$ contains a prime ideal of $N(R)$).

Next, I examine consequences of the assumption that $mI$ is finitely generated. For any ideal $I$ of $R$ let $\mathcal{I}(I)$ denote the set of
finitely generated ideals \( F \) of \( I \) such that \( FI = F \). It is shown in [7, Theorem 76] that:

(1) If \( F \in \mathcal{F}(I) \), there is an \( i \in I \) such that \( a(1 - i) = 0 \) for all \( a \in F \). That is, \( F \subseteq mI \).

Suppose \( I \) is an ideal of a ring \( R \). If \( ab \in I \) and \( a \notin I \) imply \( b \in \sqrt{I} \), the \( I \) is called a primary ideal. The radical of a primary ideal is a prime ideal [13, p. 152]. If whenever \( A \) and \( B \) are ideals of \( R \), \( AB \subseteq I \), and \( A \notin I \) imply \( B^n \subseteq I \) for some positive integer \( n \), then \( I \) is called a strongly primary ideal. It is known that a primary ideal with finitely generated radical is strongly primary [13, p. 200, proof of 2)].

Let \( I^n = \bigcap_{i=1}^{n} I^n \), and note that if \( azmI \), there is an \( i \in I \) such that \( a = ai = ai^2 = \ldots = ai^n \) for every positive integer \( n \). Thus \( mI \subseteq I^n \).

16. Proposition. Suppose \( I \) is an ideal of a ring \( R \).

(a) If \( mI \) is finitely generated, then \( mI \) is the largest element of \( \mathcal{F}(I) \) and \( mI = A(1 - i) \) for some \( i \in I \).

(b) If \( I^n \) is finitely generated, then \( mI = I^n \) if and only if \( I^mI = I^n \).

(c) If \( I^n \) is finitely generated and \( I^mI \) is an intersection of strongly primary ideals, then \( mI = I^n \).

(d) If \( R \) is Noetherian, then \( mI = I^n \).

Proof. Since \( (mI)I = mI \), (a) follows from (1), and (b) follows from (a) and the fact that \( mI \subseteq I^n \).

Suppose \( I^mI \) is contained in a strongly primary ideal \( Q \). If \( I \notin \sqrt{Q} \), then \( I^n \subseteq Q \) since \( Q \) is primary. If \( I \subseteq \sqrt{Q} \), then there is a positive integer \( n \) such that \( I^n \subseteq I^n \subseteq Q \), since \( Q \) is strongly primary. Hence \( I^mI = I^n \) and (c) follows from (b).

Finally (d) follows from (c) since every ideal of a Noetherian ring is an intersection of (strongly) primary ideals [11, p. 199].

Proposition 16 (d) is also proved in [12, p. 49].

The next two examples show that some of the assumptions made in Proposition 16 (c) are necessary.
17. **Example.** An integral domain $D_1$ such that if $M$ is a maximal
ideal of $D_1$ then $M^\omega M = J(D_1)$ is a prime ideal, but $mM \neq M^\omega$.

Let $D_1$ denote the ring of formal power series $a(x) = \sum_{n=0}^{\infty} a_n x^n$
with rational coefficients such that $a(0) = a_0$ is an integer. As is
noted in [3, p. 162], $M$ is a maximal ideal of $D_1$ if and only if there
is a prime integer $p$ such that $M_1 = pD_1$. Moreover $(pD_1)^\omega =
= \{a(x) \in D_1 : a(0) = 0 \} = J(D_1)$, and, clearly $(pD_1)^\omega(pD_1) = (pP_1)^\omega$.
Since $D_1$ is an integral domain $m(pD_1) = \{0\} \neq (pD_1)^\omega$. Note that
$(pD_1)^\omega$ is not finitely generated since for $n = 0, 1, 2, \ldots, \left( \frac{1}{2^n} x \right) D_1$ is a
strictly ascending chain of ideals contained in $(pD_1)^\omega$.

18. **Example.** An integral domain with a prime ideal $P$ such
that $P^\omega$ is both prime and principal, but $mP \neq P^\omega$.

If $D_1$ is the ring of Example 17, let $D_2 = D_1 [[y]]$ denote the
ring of formal power series with coefficients in $D_1$. Let

$$P = \{a(y) = \sum_{n=0}^{\infty} a_n(x) y^n : a_0(x) \in D_1 \text{ for } n \geq 0 \text{ and } a_0(x) \in J(D_1)\}.$$ 

Thus $a(y) \in P$ if and only if when we write $a_n(x) = \sum_{k=0}^{\infty} a_{nk} x^n$, we
have $a_{00} = 0$. It is easily verified that $P$ is a prime ideal, and $P^\omega = \{a(y) \in D_2, a(0) = 0\}$ is also a prime ideal. Since $D_2$ is an
integral domain, $mP = \{0\} \neq yD_2 = P^\omega$. Note finally that
$yP^\omega = yP \cap P^\omega = yP^\omega = P^\omega$ is a prime ideal, but, by Proposition
16, $PP^\omega$ is not an intersection of strongly primary ideals.

The next proposition provides another sufficient condition for
$J(R)$ to contain a prime ideal.

19. **Proposition.** Suppose $P$ is a minimal prime ideal of a
ring $R$ such that

(i) $P$ is finitely generated, and

(ii) there is a maximal ideal $M \supset P$ and an ideal $B$ of $R$ for
which $P = MB$.

Then:

(a) $\sqrt{mP} = P$ if $P = M$ and $mM = P$ if $P \neq M$.

(b) If $J(R)$ is pseudoprime, then it contains a unique minimal
prime ideal of $R$.

**Proof.** If $P = M = MR$, then (a) holds by Proposition 3. If $P \neq M$, then $B \subset P$ since $P$ is prime, and $P = MB \subset MP \subset P$. Thus
\( P = MP \), so \( P \subseteq mM \) by (1) and \( mM \subseteq P \) by Proposition 3. Hence \( P = mP \) and (a) holds in this case as well.

Part (b) follows from (a) and Theorem 7.

An ideal \( B \) of a ring \( R \) is called a multiplication ideal if whenever \( A \) is an ideal of \( R \) such that \( A \subseteq B \), there is an ideal \( C \) of \( R \) such that \( A = BC \). If every ideal of \( R \) is a multiplication ideal, then \( R \) is called a multiplication ring. The ring \( R \) is called an almost multiplication ring if every ideal with a prime radical is a power of its radical. The following facts are known.

(2) Every multiplication ring is an almost multiplication ring and every Noetherian almost multiplication ring is a multiplication ring \([10, \text{p. 216 and p. 213, Theorem 9.21}]\).

(3) If \( P \) is a prime ideal and \( M \) is a maximal ideal of an almost multiplication ring such that \( P \subseteq M \) and \( P \neq M \), then \( P = MP \). \([10, \text{p. 224, Ex. 9}]\)

With the aid of (2) and (3) the following consequences of Proposition 19 follow.

20. \textbf{Theorem.} If the Jacobson radical \( J(R) \) of a ring \( R \) is a pseudoprimed multiplication ideal and if every radical ideal of \( R \) contained in \( J(R) \) is finitely generated, then \( R \) is an integral domain or \( J(R) \) is a minimal prime ideal of \( R \). In particular, the Jacobson radical of a Noetherian (almost) multiplication ring contains a unique minimal prime.

\textbf{Proof.} By Proposition 19 and (3), \( J(R) \) contains a unique minimal prime ideal \( P \). Since \( J(R) \) is a multiplication ideal, if \( P \neq J(R) \) there is an ideal \( B \) of \( R \) such that \( P = J(R)B \). Since \( P \) is prime, \( B \subseteq P \), so \( P = J(R)B \subseteq J(R)P \subseteq P \), and hence \( P = J(R)P \). Hence by (1) and Lemma 1, \( P \subseteq mJ(R) = \{0\} \). Thus \( R \) is an integral domain. This completes the proof of the theorem.

The next example shows that a Noetherian ring may have a pseudoprimed Jacobson radical which contains no prime ideal.

21. \textbf{Example.} A semiprimed Noetherian ring \( R \) with pseudoprimed Jacobson radical \( J(R) \) which has exactly three minimal prime ideals, none of which are in \( J(R) \). If \( F \) is any field, let \( T = F[x_1, x_2, x_3] \) denote
the ring of polynomials in three indeterminates $x_1, x_2, x_3$. Let $I = x_1 x_3 T + x_1 x_2 T + x_2 x_3 T$, and let $T^e = \left\{ \frac{a}{1 - i} : a \in T, i \in I \right\}$ denote the quotient ring of $R$ with respect to the multiplicative system $\{1 - i : i \in I\}$. Finally, let $R = T^e/(x_1 x_2 x_3)T^e$, and let $b = b + x_1 x_2 x_3 T^e$ for any $b \in T^e$.

Since $T$ is a Noetherian unique factorization domain, $R$ is Noetherian, and each of its proper divisors of $0$ is a multiple of $\overline{x}_1, \overline{x}_2$, or $\overline{x}_3$. Clearly, also, $I = \overline{x}_1 \overline{x}_2 R + \overline{x}_2 \overline{x}_3 R + \overline{x}_3 \overline{x}_1 R \subset J(R)$, and it follows that $J(R)$ is pseudoprime. Since every element of a minimal prime ideal is a proper divisor of $0$, the minimal prime ideals of $R$ are $P_i = \overline{x}_i R$ for $i = 1, 2, 3$, none of which are contained in $J(R)$ since $1 - \overline{x}_i$ is not a unit of $R$. Finally, $R$ is semiprime because $P_1 \cap P_2 \cap P_3 = \{0\}$.

In view of Example 22, the following proposition may not seem so special.

22. **Proposition.** If $R$ is a ring with no more than two minimal prime ideals and $J(R)$ is pseudoprime, then $J(R)$ contains a prime ideal.

**Proof.** If $R$ has exactly one minimal prime ideal, it must be $N(R) \subset J(R)$. Suppose $R$ has two minimal prime ideals $P_1, P_2$. By Remark 15(b), we may assume that $R$ is semiprime. By Proposition 3, if $M \in M(R)$, then $mM$ is $P_1, P_2$, or $P_1 \cap P_2 = \{0\}$. Hence $S(R) = \{0\}$, or $S(R)$ contains a prime ideal. In the first case, the conclusion follows from Theorem 10, and in the second case it follows from Theorem 7.

I conclude with an example that shows that the hypothesis of Proposition 22 can be satisfied for a ring $R$ without $R$ being presimplifiable.

23. **Example.** A semiprime Noetherian ring $R$ with two minimal prime ideals such that $J(R) \not\subset\mathfrak{P}(R)$ and $R$ is not presimplifiable. Let $S$ denote the ring of formal power series with $0$ constant term with coefficients from the ring of integers mod $2$. Clearly $S$ is Noetherian and $J(S) = S$. If $Z$ denotes the ring of integers, let $R = S^* Z = \{ (a, n) : a \in S, n \in Z \}$ where for $a_1, a_2 \in R, n_1, n_2 \in Z, (a_1, n_1) + (a_2, n_2) = (a_1, n_1) + (a_2, n_2)$ and $(a_1, n_1)(a_2, n_2) = (a_1 a_2 + n_2 a_1 + n_1 a_2, n_1 n_2)$. It is well known that $R$ is a Noetherian ring with identity and the mapping
a → (a,0) is an injection of S onto a prime ideal $\mathfrak{s}$ of R. It is easily verified that $\mathfrak{s} = \mathcal{J}(R)$. Also since $(a,0)(0,2) = (0,0)$ for every $as,\mathcal{J}(R) = \mathfrak{s}$ is a minimal prime ideal of R. By the same reasoning $P = \{(0,2n) : n \in \mathbb{Z}\} \in \mathcal{P}(R)$, and any other prime ideal of R contains a regular element. So $\mathcal{P}(R) = \{\mathcal{J}(R),P\}$, and R is not presimplifiable since $P \notin \mathcal{J}(R)$. Finally, R is semiprime since $P \cap \mathcal{J}(R) \subset \{0\}$.

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