TYCHONOFF SPACES THAT HAVE A COMPACTIFICATION
WITH COUNTABLE REMAINDER

MELVIN HENRIKSEN

Claremont

In 1935, L. Zippin showed that every separable rimcompact completely metrizable space has a metrizable compactification with a countable (not necessarily infinite) remainder [2]. A Tychonoff space $X$ with a compactification $γX$ such that $|γX - X| ≤ ω$ is called a Zippin space and $γX$ is called a Zippin compactification. If, in addition, $γX - X$ is metrizable, $X$ is called a strongly Zippin space and $γX$ a strongly Zippin compactification. In this paper, an attempt is made to characterize spaces that are Zippin or strongly Zippin.

We succeed in this goal only in small part, but we do obtain a number of conditions on a space that are either necessary or sufficient for such compactifications to exist. For the most part, proofs are omitted. A more complete version of this paper will appear elsewhere.

At the Fourth Prague Topological Symposium, T. Hosokawa also presented a paper on this topic. His results and mine overlap, but are not identical.

All topological spaces considered are assumed to be Tychonoff spaces. Any such space has a maximal compactification $δX$, called the Stone-Cech compactification of $X$ that maps continuously onto any compactification $γX$ of $X$ with a mapping that extends the identity map [GJ, Chapter 6]. If the topology of $X$ has a base of open sets with compact boundary, then $X$ is called rimcompact (the term semicompact is used in [2] and semibicompact is used in [M]). Every rimcompact space has a compactification $δX$ maximal among the compactifications with a zero-dimensional remainder. $δX$ is called the Freudenthal compactification of $X$ [I, pp. 109-122] [M].

If $P$ is a property of topological spaces, then $X$ has $P$ at $σ$ if $δX - X$ has $P$. It is noted in [HI, Sec. 3] that if $P$ is compactness, local compactness, or the Lindelöf property, then $X$ has $P$ at $σ$ if and only if $γX - X$ has $P$ for any compactification $γX$ of $X$. A space that is $σ$-compact at $σ$ is said to be Čech-complete or an absolute $G_δ$. It is well known that a metrizable space is Čech-complete if and only if it admits a complete metric [E, p. 190]. $X$ is Lindelöf at $σ$ if and only if every compact subset $K_1$ of $X$ is contained in a compact set $K_2$ for which there is a countable family of open sets containing $K_2$ such that any open set containing $K_2$ contains some $U_1$. In particular, every metrizable space is Lindelöf at $σ$ [HI, Sec. 3]. Also, if $X$ is Lindelöf at $σ$ and has a compactification with 0-dimensional
remainder, then $X$ is rimcompact by [I, p. 114].

It follows that every Zippin space is rimcompact and Čech-complete. (See also [R1] [R2]). As is noted in [I, p. 109]:

$$\text{Cl}_{\gamma X}(\gamma X - X) = \gamma X - X \cup R(X)$$

for any compactification $\gamma X$ of $X$,

where $R(X)$ is the set of points of $X$ that fail to have a compact neighborhood.

Thus, by [CN, Sec. 6], we have:

1. **Proposition** If $X$ is a Zippin space then
   
   (a) $X$ is rimcompact.
   
   (b) $X$ is Čech-complete.
   
   (c) $|R(X)| \leq \exp \exp u$.
   
   If $X$ is strongly Zippin, then, in addition:
   
   (d) $R(X)$ is a Lindelöf space.

   The upper bound in (c) cannot be lowered. For if $Q$ is the space of rational numbers, then $Q = \gamma Q$ is a strongly Zippin compactification of $\gamma Q = R(\gamma Q)$, and $|\gamma Q| = |\gamma Q - Q| = \exp \exp u$ [GJ, Chap. 9].

   Whether the conditions of Proposition 1 are sufficient to insure that a space $X$ is a Zippin space remains an open question. Below, two kinds of sufficient conditions are obtained; those that make $R(X)$ a "large" part of $X$, and those that make it in a sense "small". I begin with the former.

   A space $X$ such that every family of pairwise disjoint open sets is countable is said to satisfy the countable chain condition (CCC). A space $X$ is called metacompact or weakly paracompact if every open cover has a point-finite open refinement. As is well known, every paracompact, and hence every metrizable space is metacompact [E, pp. 225-228].

   As in [JM], a space $X$ is called dense separable if every dense subspace of $X$ is separable.

2. **Theorem.** Suppose $X$ is a Zippin space such that $X - R(X)$ is separable. Then:

   (a) $X$ satisfies the CCC.
   
   (b) If $X$ is metacompact or strongly Zippin, then $X$ is a Lindelöf space.
   
   (c) If $X$ is strongly Zippin, then $X$ is separable.
   
   (d) If $X - R(X)$ is dense separable, so is $X$. 
3. Corollary. Suppose $X$ is a metrizable space such that $(X-R(X))$ is separable. Then the following are equivalent.

(a) $X$ is a strongly Zippin space.
(b) $X$ is a Zippin space.
(c) $X$ is separable, rimcompact, and Čech-complete.

Next, a characterization of a special class of strongly Zippin spaces is given. It is established by decomposing the remainder of $X$ in its Freudenthal compactification $\beta X$.

4. Theorem. If $R(X)$ is locally compact, then $X$ is a strongly Zippin space if and only if $X$ is rimcompact, Čech-complete, and $R(X)$ is a Lindelöf space. Indeed, such a space has a strongly Zippin compactification with remainder homeomorphic to either a countable discrete space or its one-point compactification.

I conclude with some remarks, examples, and questions.

A. By modifying [LM, Example 5.3], an example can be given of a Zippin space that is not strongly Zippin. It can be shown, however, that if $R(X)$ is Lindelöf and $X$ is a Zippin space, then $X$ is strongly Zippin.

B. Clearly every closed subspace of a (strongly) Zippin space is (strongly) Zippin, and every open subspace of a Zippin space is rimcompact and Čech-complete by Proposition 1. The existence of open subspaces of $\beta\mathbb{Q}$ that are not Lindelöf shows that an open subspace of a strongly Zippin space need not be strongly Zippin. I do not know, however, if an open subspace of a (strongly) Zippin space has to be a Zippin space.

C. Recall that a continuous closed surjection $f: X \to Y$ such that $f^{-1}(y)$ is compact for every $y \in Y$ is called a perfect map. If $Y = [0,1]-\mathbb{Q}$, then the projection map of $Y \times [0,1]$ onto $Y$ is perfect, $Y$ is a strongly Zippin space, but $\mathbb{Q} \times [0,1]$ is not rimcompact and hence is not a Zippin space (although it is the product of a compact space and a strongly Zippin space). I do not know, however, if a perfect image of a (strongly) Zippin space must be (strongly) Zippin.

D. It follows easily from [LM, Example 5.3, ff.] that no connected Zippin space has a countable partition into compact sets.

E. It is easily verified that if $R(X) = X$ is connected, then the remainder of $X$ in any compactification is connected, whence $X$ cannot be a Zippin space. (See [R 1, Corollary 3]). Indeed, if $X$ is also Lindelöf at $\infty$, it cannot even be rimcompact. In particular, a countably infinite product of copies of $\mathbb{R}$ is not rimcompact.

F. It was shown by McCartney in [Mc, 3.6] that $X$ has a maximal Zippin compactification if and only if $X$ has a compactification with zero-dimensional remainder.
and $8X$ has only countably many components. Indeed, if this latter holds, then $4X$ is the maximal Zippin compactification. For a simpler proof see [D].

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References


Harvey Mudd College