SOME PROPERTIES OF POSITIVE DERIVATIONS ON f-RINGS

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1. INTRODUCTION

Throughout A denotes an f-ring; that is, a lattice-ordered ring that is a subdirect union of totally ordered rings. We let \( D(A) \) denote the set of derivations \( D : A \rightarrow A \) such that \( a \geq 0 \) implies \( Da \geq 0 \), and we call such derivations positive. In [CDKJ], P. Coleville, G. Davis, and K. Keimel initiated a study of positive derivations on f-rings. Their main results are (i) \( D \in D(A) \) and \( A \) archimedean imply \( D = 0 \), and (ii) if \( A \) has an identity element 1 and \( a \) is the supremum of a set of integral multiples of 1, then \( Da = 0 \). Their proof of (i) relies heavily on the theory of positive orthomorphisms on archimedean f-rings and gives no insight into the general case. Below, in Theorem 4 and its corollary, we give a direct proof of (i), and in Theorem 10, we generalize (ii). Throughout, we improve on results in [CDKJ], and we study a variety of topics not considered therein.

2. THE RESULTS

In the sequel, \( A \) will always denote an f-ring, and \( A^+ = \{ a \in A : a \geq 0 \} \) its positive cone. If \( a \in A \), let \( a^+ = a \vee 0 \), \( a^- = (-a) \vee 0 \), and \( |a| = a \vee (-a) \). Then \( a = a^+ - a^- \), \( |a| = a^+ + a^- \), and \( a^2 = a^- a^+ = a^+ a^- = 0 \). A subset I of \( A \) that is a ring ideal and such that \( |b| \leq |a| \), and \( a \in I \) imply \( b \in I \) is called an \( \ell \)-ideal. The \( \ell \)-ideals are the kernels of homomorphisms that preserve lattice as well as ring operations [BK, Chap. 8].

A derivation on \( A \) is a linear map \( D : A \rightarrow A \) such that if \( a, b \in A \), then \( D(ab) = abD + (Da)b \). A derivation \( D \) is called positive if \( D(A^+) \subseteq A^+ \). The family of all positive derivations on \( A \) will be denoted by \( D(A) \).
In any f-ring \( \text{rad} A \), the set of all nilpotent elements of \( A \), coincides with the intersection of all the prime \( \mathfrak{I} \)-ideals of \( A \), and hence is an \( \mathfrak{I} \)-ideal [BKW, 9.2.6]. If \( \text{rad} A = \{0\} \), then \( A \) is said to be reduced. In [CCDJ], it is shown that if \( A \) is commutative and \( a^n = 0 \), then \( (Da)^{2n-1} = 0 \). We improve this result next. We begin by observing that if \( a, b, c \in A^+ \) then

\[(1) \quad ab = 0 \implies aDb = (Da)b = 0.\]

1. **PROPOSITION.** Suppose \( a \in A \) and \( D \in D(A) \). Then \( a^n = 0 \) implies \( (Da)^n = 0 \). In particular, \( D\text{rad} A \subseteq \text{rad} A \).

**PROOF.** Since \( a^n = 0 \) if and only if \( |a|^n = 0 \), we may assume \( a \in A^+ \) and \( n > 1 \). By (1), \( a^{n-1}Da = 0 \). So \( a^{n-2}(aDa) = 0 \). Using (1) again yields \( 0 = a^{n-2}D(aDa) = a^{n-1}Da + a^{n-2}(Da)^2 \). Since \( a \in A^+ \), \( a^{n-2}(Da)^2 = 0 \). Continuing this process yields \( (Da)^n = 0 \) and hence that \( D\text{rad} A \subseteq \text{rad} A \).

The next example will show that the index of nilpotency of \( Da \) need not be less than that of \( a \). We note first that if \( D \in D(A) \) and \( I \) is an \( \mathfrak{I} \)-ideal of \( A \) such that \( D(I) \subseteq I \), then \( D_I \in D(A/I) \), where

\[D_I(a+I) = Da+I,\]

2. **EXAMPLE.** Let \( R \) denote all rational functions with real coefficients of negative degree. If \( r(x) = \frac{p(x)}{q(x)} \in R \), we may assume that \( q(x) = x^m + a_1x^{m-1} + \ldots \) has leading coefficient \( 1 \), and we let \( r(x) \) be positive if the leading coefficient of \( p(x) \) is positive. With this order, \( R \) is a totally ordered ring. If \( r(x) \in R \), let \( Dr(x) = -r'(x) \) be the negative of the usual derivative. Then \( D \in D(R) \), as is \( (xD): R \rightarrow R \), where \( (xD)r(x) = xDr(x) = -xr'(x) \). If \( n \) is a positive integer, let \( I_n \) denote the set of all \( r(x) \) in \( R \) of degree \( \leq -n \). Clearly \( I_n \) is an \( \mathfrak{I} \)-ideal of \( R \), and \( (xD)(I_n) \subseteq I_n \). If \( R_n = R/I_n \), and \( (xD)_n(r(x)+I_n) = xDr(x) + I_n \), then \( (xD)_n \in D(R_n) \), and \( (xD)_n(\frac{1}{x}+I_n) = \frac{1}{x} + I_n \) is nilpotent of index \( n \).

If \( G \) is an abelian \( \mathfrak{I} \)-group, and \( T: G \rightarrow G \) is an order preserving endomorphism of \( G \) such that \( x \wedge y = 0 \) implies \( x \wedge Ty = 0 \) for \( x, y \) in \( G^+ \), then \( T \) is called a **positive orthomorphism** of \( G \). If \( A \) is reduced, then \( x \wedge y = 0 \) if and only if \( xy = 0 \) [BKW, 9.3.1].
So each positive derivation on an f-ring is an orthomorphism by (1). The next result appears implicitly in [CDKJ]. We include a proof for the sake of completeness.

3. PROPOSITION. If $P$ is a minimal prime $\ell$-ideal of $A$, and $D \in D(A)$, then $D(P) \subseteq P$. In particular, $D_P \subseteq D(A/P)$.

PROOF. As is noted in [BK, 9.3.2 and 12.1.11], if $A$ is reduced, then each positive orthomorphism of $A(\mathbb{P})$ maps a minimal prime subgroup into itself, and $P$ is a minimal prime $\ell$-ideal of $A$ if and only if it is a minimal prime subgroup. So $D(P) \subseteq P$ if $A$ is reduced. In the general case, if we let $I = \text{rad } A$ in (2), we obtain $D(P) \subseteq P$.

We do not know if $D(P) \subseteq P$ for any prime $\ell$-ideal of $P$.

Recall that $A$ is said to be archimedean if $a \in A^+$ and $(na : n=1,2,\ldots)$ bounded above imply $a = 0$. The next theorem is the key to an alternate proof of the fact that a reduced archimedean f-ring admits no nontrivial derivations [CDKJ].

4. THEOREM. Suppose $A$ is reduced, $D \in D(A)$, $a \in A^+$, and $n$ is a positive integer. Then

(a) $n(a \cdot a^2)D_a \leq (a \cdot a^2)D_a$,

(b) $nD_a(a \cdot a^2) \leq D_a(a \cdot a^2)$, and

(c) $nD(a^2) \leq (a^2D_a + (Da)a^2) \vee Da$.

PROOF. Since $A$ is reduced, $(0)$ is an intersection of minimal prime ideals and $A$ is a subdirect sum of totally ordered rings $A/P$ such that $P$ is a minimal prime $\ell$-ideal. Thus, by Proposition 3, it suffices to verify these identities in case $A$ is totally ordered and has no proper divisors of $0$ [BK, 9.2.5]

Let $x = (na-a^2)^+D_a$. Then $x \in A^+$. We consider two cases:

(i) Suppose $x = 0$. Then $Da = 0$ or $na \leq a^2$. In either case we obtain

$$nD(a^2) \leq (a^2D_a + (Da)a^2) \vee Da$$

(ii) Suppose $x > 0$. Then $Da > 0$ and $a^2 < na$. Hence $aD_a + (Da)a \leq nD_a$. Since $A$ is totally ordered, $aD_a \leq (Da)a$ or $(Da)a \leq a(Da)$.

Suppose the former holds. Then

$$2aD_a \leq nD_a$$

and hence $(na-2a^2)Da \geq 0$.  

But $Da > 0$, so $2a^2 \leq na$. By induction, we get $2^k a^2 \leq na$ for $k = 0, 1, 2, \ldots$. If we choose $k$ so large that $n^2 \leq 2^k$, we get

$$na^2 \leq a.$$  

If, instead, $(Da)a \leq aDa$, an obvious modification of this latter argument also yields (4). Pre or post multiplying by $Da$ yields

$$na^2 Da \leq aDa \quad \text{and} \quad n(Da)a^2 \leq (Da)a.$$  

Since either (3) or (5) must hold in $A/P$ for any minimal prime ideal $P$, the conclusions of (a) and (b) hold.

By (4), if $x > 0$, then $nD(a^2) \leq D(a)$. If $x = 0$, then adding the inequalities in (3) yields $nD(a^2) \leq (a^2 Da + (Da)a^2)$. Hence (c) holds as well.

5. COROLLARY. [CDK] If $A$ is archimedean and $D \in D(A)$, then $D(A) \subseteq \text{rad } A$ and $D(A^2) = 0$.

PROOF. By (c) of the last theorem and Proposition 3, if $a \in A$, then $D(a^2) \subseteq \text{rad } A$. Since $aDa \leq D(a^2)$, $(Da)^2 \leq D(aDa) \leq D^2(a^2) \subseteq D(\text{rad } A) \subseteq \text{rad } A$ by Proposition 1. Since each element of $\text{rad } A$ is nilpotent, so is $Da$.

If $a, b \in A$, then $D(ab) = aDb + (Da)b = 0$, since $(\text{rad } A)A = A(\text{rad } A) = 0$ in an archimedean f-ring [BKW, 12.3.11]. Hence $D(A^2) = 0$.

5. PROPOSITION. Suppose $e^2 = e \in A$ and $D \in D(A)$.

(a) $((De))^2 = e(De)e = (De)(De) = 0$.

(b) If $A$ is reduced or has an identity element or $e$ is in the center of $A$, then $De = 0$.

PROOF. Since $e^2 = e$, we have

$$eDe + (De)e = De$$

Multiplying (6) on the left by $e$ yields

$$eDe = 0.$$  

Applying $D$ to (7), we obtain

$$eD(De)e + (De)e = 0 = e(De)^2 + D(eDe)e.$$
Hence

\[(8) \quad e(De)^2 = (De)^2 = 0.\]

Multiplying both sides of (6) on the left by \(De\) and using (8) yields

\[(9) \quad (De)e(De) = (De)^2.\]

By (7), (8), and (9), we obtain

\[eDe - (De)e^2 + (De)e(De) = 0.\]

Hence \((De)^2 = (De)e(De) = 0\), which together with (7), completes the proof of (a).

Clearly \(De = 0\) if \(\text{rad } A = \{0\}\). If \(eDe = (De)e\), then by (6) and (7), \(De = 2eDe = 0\). If \(A\) has an identity element, then each of its idempotents is in the center of \(A\) by [BKWN, 9.4.201]. This completes the proof of (b).

The next example shows that the hypotheses of (b) above cannot be omitted.

7. EXAMPLE. A totally ordered ring with an idempotent \(e\) and a positive derivation \(D\) such that \(De \neq 0\).

Let \(S\) denote the algebra over the real field \(\mathbb{R}\) (with the usual order) with basis \(\{e, z\}\), where \(e^2 = e\), \(ez = z^2 = 0\), and \(ze = z\). If \(x = ae + bz \in S\), let \(x > 0\) if \(a > 0\) or \(a = 0\) and \(b > 0\). If we let \(Dx = z - xz = az\), then \(D \in \mathcal{D}(S)\), and \(De = z \neq 0\).

If \(D \in \mathcal{D}(A)\), let \(\ker D = \{a \in A : Da = 0\}\). If \(G\) is an abelian \(\ell\)-group and \(H \subseteq G\), let \(H^\perp = \{g \in G : |g| \cdot |h| = 0\ \text{for all } h \in H\}\), and let \(H^{\perp\perp} = (H^\perp)^\perp\). Note that \(H^\perp\) is an \(\ell\)-subgroup of \(G\) (that is, \(H\) is a subgroup and \(|a| \leq |b|\), and \(b \in H^\perp\) implies \(a \in H\)). A band in \(G\) is an \(\ell\)-subgroup \(H\) of \(G\) such that if \(K \subseteq H\) and \(\sup K \in G\), then \(\sup K \in H\). If \(H\) is a subset of \(G\), the intersection \(B(H)\) of all the bands in \(G\) containing \(H\) is also a band. Moreover, \(B(H) \subseteq H^{\perp\perp}\). See [ELZ, Theorem 19.2]. An element \(e\) of \(G\) such that \((e)^\perp = 0\) is called a weak order unit of \(G\). An element \(a\) of an \(\ell\)-ring \(A\) such that \(ex = 0\) or \(xe = 0\) implies \(x = 0\) is called regular. Note that if \(e \in A\) is regular, then \(e\) is a weak order unit, and the converse holds if \(A\) is reduced.
The following lemma will be useful in what follows.

8. **Lemma.** Suppose \( A \) is an \( \ell \)-ring and \( D \in D(A) \).

(a) \( xDx \leq (Dx)x \leq 0 \) for every \( x \in A \).

(b) If \( A \) is reduced, then \( D \) is an \( \ell \)-endomorphism.

(c) If \( A \) has an identity element \( 1 \), and \( n \) is a positive integer, then \( nDx \leq xDx \leq (Dx)x \) for every \( x \in A^+ \) and \( D(1) = 1 \) for every \( \ell \)-ideal \( I \) of \( A \).

**Proof.** (a) holds since this inequality holds whenever \( A \) is totally ordered.

(b) holds since if \( A \) is reduced, then \( D \) is a positive orthomorphism and hence an \( \ell \)-endomorphism [BK1, 12.1].

(c) by Proposition 6(b), \( 1 \in \ker D \), and by (a) \( (x-n1)D(x-n1) \geq 0 \).

Hence \( nDx \leq xDx \). Similarly, \( nDx \leq (Dx)x \). Hence \( x \in I \) implies \( Dx \in I \) since \( I \) is an \( \ell \)-ideal.

Next, we provide some examples to show that the hypotheses of (b) and (c) above cannot be omitted.

9. **Examples.** (i) Let \( E \) denote the direct sum of two copies of the real line \( \mathbb{R} \) with trivial multiplication, and let \( (r,s) \geq 0 \) mean \( r \geq s \geq 0 \). As is noted in EG1, 5B2, the map \( D: E \to E \) such that \( D(r,s) = (r,0) \) is a positive endomorphism that is not an \( \ell \)-homomorphism.

To see the latter, note that \( (1,2)^+ = (2,2) \). So \( D(1,2)^+ = (2,0) \neq (1,0) = E(1,2)^+ \).

(ii) Let \( R \) and \( (xD) \) be as in Example 2, and let \( y = \frac{1}{x} \). Then \( n(xD)y = \frac{n}{x} \), while \( y(xD)y = x^{-2} \), so the conclusion of (c) fails.

The next theorem summarizes most of what we know about kernels of positive derivations.

10. **Theorem.** Suppose \( D \in D(A) \), \( x \in A \), and \( n \) is a positive integer.

(a) If \( x \) is regular, and \( ex \in \ker D \), then \( x \in \ker D \).

(b) If \( A \) is reduced then:

(i) \( x \in \ker D \) implies \( (x)^n \in \ker D \),

(ii) \( x^n \in \ker D \) implies \( x \in \ker D \),

(iii) \( \ker D \) is a band.
(iv) \( D^n = 0 \) implies \( D = 0 \), and
(v) \( e^2 = e \in A \) implies \( e \in \ker D \).

(c) If \( A \) has an identity element and \( U(A) \) is the smallest band containing the units of \( A \), then \( U(A) \subseteq \ker D \). In particular, \( \text{rad} A \subseteq \ker D \). Also, if \( x^2 \leq x \), then \( x \in \ker D \).

PROOF. (a) By (1), \( D(\text{ex}) = 0 \) implies \( eDx = 0 \), which, in turn implies \( Dx = 0 \).

(b) (i) By Lemma 8(b), and [BKW, 3.2.3], \( D((x)^{\perp}) \subseteq D((x)^{\perp} \perp) \perp \subseteq (Dx)^{\perp} \perp = \{0\} \) since \( x \in \ker D \) and \( A \) is reduced.

(ii) follows from (i) and the fact that \( (x)^{\perp} \perp \) is the intersection of all the minimal \( \ell \)-ideals that contains \( x \) [BKW, 3.4.12].

(iii) As was noted above, the smallest band containing \( \ker D \) is contained in \( ((\ker D)^{\perp})^{\perp} \) and the latter is contained in \( \ker D \) by (i).

(iv) Since \( x \) is a difference of positive elements, it suffices to show that \( Dx = 0 \) whenever \( x \in A^+ \). The proof will proceed by induction on \( n \). It is obvious when \( n = 1 \). Assume that \( D^n(A) = 0 \) implies \( D(A) = 0 \) whenever \( A \) is a reduced \( f \)-ring and \( n \geq 1 \) is an integer. If \( 0 = D^{n+1}(A) = D^n(D(A)) \), then \( D^n(A^{\perp}) = 0 \) by (i).

So \( D(D(A)^{\perp}) = 0 \) by the induction hypothesis. In particular, \( D^2(x^2) = 0 \).

Since \( xDx \leq D(x^2) \), \( D = D(xDx) = xD^2x + (Dx)^2 \). So \( (Dx)^2 = 0 = Dx \) since \( A \) is reduced.

(v) is a restatement of Proposition 16(b).

(c) That \( U(A) \subseteq \ker D \) follows directly from (a) and (b) (iii) above. If \( x^n = 0 \), then \( (1-x)(1-x+n-1) = 1 \), so \( 1-x \) is a unit and \( x = 1 - (1-x) \in U(A) \subseteq \ker D \). Finally, if \( x^2 \leq x \), then \( D(x^2) = xDx + (Dx)x \leq Dx \leq xDx \perp (Dx)x \) by Lemma 8(c). Hence \( xDx = (Dx)x = 0 \). Thus \( Dx = 0 \). This completes the proof of Theorem 10.

11. EXAMPLES AND REMARKS. The assumption that \( A \) is reduced in Theorem 10(b) cannot be dropped. For example, if \( A = \mathbb{C}[0,1] \), the \( \ell \)-group of continuous real-valued functions on \( [0,1] \), with trivial multiplication for all \( f \in \mathbb{C}[0,1] \), we let \( Df = f(\frac{1}{2}) \), then \( D \in D(A) \), and \( \ker D \) fails to be a band [DV, p. 121]. Also, the plane \( E^2 \) with the usual coordinatewise addition and trivial multiplication admits positive endomorphisms that are nilpotent. (For example, let \( T(a,b) = (0,a) \) for all \( (a,b) \in E^2 \).

Theorem 10(c) generalizes [CDK, Theorem 7] where it is shown that \( \ker D \) contains the supremum of any set of elements bounded above by some integral multiple of the identity element.
As in [5], we let \( I_0(A) = \{ a \in A : |a| < x \text{ for some } x \in A^+ \text{ and } n = 1, 2, \ldots \} \). Clearly \( I_0(A) \) is an \( \ell \)-ideal and \( I_0(A) = \{0\} \) if and only if \( A \) is archimedean.

12. THEOREM. Suppose \( D \in D(A) \).
   (a) If \( A \) is reduced, then \( D(A^2) \subseteq I_0(A) \).
   (b) If \( A \) has an identity element, then \( D(A) \subseteq I_0(A) \). If, moreover, \( A \) is reduced and \( I_0(A) \subseteq U(A) \), then \( D = 0 \).

PROOF. (a) follows immediately from Theorem 4 and the fact that \( ab < (a + b)^2 \) whenever \( a, b \in A^+ \).
   (b) That \( D(A) \subseteq I_0(A) \) is a restatement of Lemma 10(c). If \( I_0(A) \subseteq U(A) \), then by Theorem 10(c), \( D^2(A) = D(U(A)) = \{0\} \). Hence if \( A \) is reduced, then \( D = 0 \) by Theorem 10(b).

13. EXAMPLES AND REMARKS.
   (a) The reader may easily verify for the \( f \)-ring \( R \) of Example 2, \( I_0(R) = I_2 \), while \( (x0)(R) = R \). So the hypothesis in Theorem 12(b) that \( A^+ \) has an identity element may not be dropped if we wish to have \( D(A) \subseteq I_0(A) \).
   (b) Let \( S \) denote the ring of all functions of the form

   \[
   f(x) = \sum_{i=0}^{\infty} a_i x^{r_i},
   \]

   where \( a_i \) is an integer and \( r_i \) is a nonnegative rational number, ordered lexicographically, with the coefficient of the largest power of \( x \) dominating. Then \( I_0(S) = S \), and \( U(A) \) is the set of constant polynomials. So, the condition of Theorem 12(b) fails. Despite this, \( D \in D(S) \) implies \( D = 0 \).

   For if \( D \in D(S) \), then \( D(x) = D(x^{1/2})^2 = 2x^{1/2}D(x^{1/4})^2 \)
   \[ = 4x^{3/4}D(x^{1/8})^2 = \ldots = 2^n x^{1-1/2n}D(x^{1/2n}) \]
   Hence \( 2^n | D(x) \) for \( n = 0, 1, 2, \ldots \). Since the coefficients of any element of \( S \) are integers, it follows that \( D(x) = 0 \). A similar argument will show that \( x^r \in \ker D \) whenever \( r \) is a nonnegative rational number. It follows that \( D = 0 \).

   We do not, however, know of any such example that is an algebra over an ordered field. If \( S^* \) is the result of allowing the coefficients of the elements of \( S \) to be arbitrary rational numbers, and we let \( D(x^r) = rx^r \) for any positive rational number \( r \), then \( D \) is a positive derivation. To see why, map \( x^r \) to \( e^{rx} \) and note that \( S^* \) is isomorphic as an ordered ring to a subring of the ring of exponential polynomials, and the usual derivative on the latter maps the image of \( S^* \) into itself.
Our last result applies more general theorems and techniques of Herstein \(H_j, H_2\) to the context of positive derivations.

14. **THEOREM.** Suppose \(A\) is reduced and \(D \in D(A)\).
   (a) If \(D \neq 0\), then the ring \(S\) generated by \(\{Da : a \in A\}\) contains a nonzero ideal of \(A\).
   (b) If \(S\) is commutative, then \(S\) is contained in the center of \(A\).
   (c) If \(z \in A\) commutes with every element of \(S\), \((az-za) \in \ker D\) for every \(a \in A\). If, in addition, \(A\) is totally ordered and \(D \neq 0\), then \(z\) is in the center of \(A\).

**PROOF.** (a) It is shown in \(H_j\) that the conclusion holds for any derivation on any ring if \(D^3 \neq 0\). Since \(A\) is reduced, \(D^3 \neq 0\) implies \(D \neq 0\) by Theorem 10(b).
   (b) Suppose \(a \in S\) and \(x \in A\). Then
   \[
   0 = (Da)D(ax) - D(ax)(Da) = Da[Da + (Da)x] - [Da + (Da)x]Da = Da(Da)x - x(Da).
   \]
   By \(H_2, \text{Lemma 1.1.4}\), \(Da\) is in the center of \(A\).
   (c) The second statement is shown in \(H_2\), and the first follows immediately from the second and Theorem 10(b).

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