Iterative Solutions of Systems of Linear Equations Whose Coefficient Matrix is Positive Real

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1. INTRODUCTION

During the academic year 1981–82, we worked, together with a team of students and faculty\(^1\) under the auspices of the Claremont Mathematics Clinic\(^2\), on a problem in computational aerodynamics that resulted in the report [L]. The problem came from an engineering group at Lockheed-California, who were modeling the flow of air over the surface of an aircraft. They obtained large full systems of linear equations of the form \(Ax = b\), where \(A\) is a matrix with real entries such that \(x'Ax > 0\) for all nonzero real vectors \(x\), and were applying a version of successive over-relaxation (SOR). Such a matrix is called positive real in [Y] and can have complex eigenvalues, hence need not be similar to a (symmetric) positive definite matrix.

\(^1\) Besides ourselves, the team members were Professor Duane DeTemple and students Rick Castrapel, John Burnett, Doreen Fujii, and Dorothy Robinson.

\(^2\) The Mathematics Clinic enables students and faculty to work together on practical problems obtained from industrial or government sources. For a description, see [He] and [S].
Below we give some intervals of ω-values for which the SOR iteration matrix \( L_ω(A) \) has spectral radius less than 1, when \( A \) is positive real and satisfies other conditions. We supply a few pertinent examples and discuss the necessity of some of these conditions.

Two situations are considered. In Section 2, we provide sufficient conditions for the convergence of SOR that include the original matrix being "nearly symmetric" in some sense, and the SOR matrix having a dominant real eigenvalue or satisfying some other special assumption. We also provide some instances where such special assumptions are satisfied.

In Section 3, we drop the special assumptions and show that SOR must converge for a (usually) smaller interval of ω-values. Pertinent examples are given, and it is observed that while our results are conservative in nature, the bounds we provide are given in terms of easily computed \( ∞ \)-norms of matrices as opposed to earlier results that depend on 2-norms, which are difficult to compute [O].

2. SOME SIMPLE SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF SOR

Throughout, \( N \) will denote a fixed positive integer, \( \mathbb{R} \) the real field, \( \mathbb{C} \) the complex field, and \( \mathbb{R}^N \) (respectively \( \mathbb{C}^N \)) the set of \( N \)-vectors with entries from \( \mathbb{R} \) (respectively \( \mathbb{C} \)) with the usual addition and scalar multiplication. If \( z \in \mathbb{C}^N \) (written as a column vector) let \( z^* = (\bar{z}_1, \bar{z}_2, \ldots, \bar{z}_N)^T \) denote the transpose of the \( N \)-vector whose entries are the complex conjugates of the entries of \( z \). As is well known, \( \mathbb{C}^N \) is a normed vector space with respect to the norms

\[
\| z \|_2 = (z^*z)^{1/2} = \left( \sum_{i=1}^{N} |z_i|^2 \right)^{1/2}
\]

and

\[
\| z \|_∞ = \max\{|z_i| : i ≤ N\}.
\]

See, for example, [ND, p. 131] and [V, Section 1.3].

If \( M \) is a positive integer, let \( \mathcal{M}_{M,N}(\mathbb{R}) \) (respectively \( \mathcal{M}_{M,N}(\mathbb{C}) \)), denote the set of \( M \times N \) matrices with entries from \( \mathbb{R} \) (respectively \( \mathbb{C} \)), and let \( \mathcal{M}_N(\mathbb{R}) \) (respectively \( \mathcal{M}_N(\mathbb{C}) \)) abbreviate when \( N = M \). We use the usual addition and multiplication (when defined). If \( A \in \mathcal{M}_{M,N}(\mathbb{C}) \) and \( z \in \mathbb{C}^N \), let \( \| A \|_∞ = \max\{|A\|_∞ : \| z \|_∞ = 1\} \). We will make use of the following
facts whose verification may be found in [ND, p. 164].

\[ (2.1) \| A \|_{\infty} = \max \left \{ \sum_{j=1}^{N} |a_{ij}| : 1 \leq i \leq M \right \}. \]

(2.2) If \( B \in \mathcal{M}_{N,K}(\mathbb{C}) \) for some positive integer \( K \), then \( \| AB \|_{\infty} \leq \| A \|_{\infty} \| B \|_{\infty} \). In particular, \( \| Az \|_{\infty} \leq \| A \|_{\infty} \| z \|_{\infty} \) if \( z \in \mathbb{C}^N \).

The next assertion follows easily from (2.2) and the definitions of \( \| z \|_2 \) and \( \| z \|_{\infty} \).

(2.3) If \( z \in \mathbb{C}^N \), \( \| z \|_2 = 1 \) and \( A \in \mathcal{M}_N(\mathbb{C}) \), then \( \| z^* Az \|_{\infty} \leq \| A \|_{\infty} \).

(2.4) Definitions If \( A \in \mathcal{M}_N(\mathbb{R}) \), then \( A \) will be called positive real if \( x^t Ax > 0 \) whenever \( x \in \mathbb{R}^N \) and \( x \neq 0 \), and \( A \) will be called positive definite if it is symmetric and positive real.

Our definitions in (2.4) agree with the usage in [Y]. Historically, most interest has been in the second, stronger property, but several authors have investigated matrices which may not be symmetric, but do have the property we call positive real. There are, however, no standard names for these concepts.\(^3\)

In what follows, \( I \) and \( 0 \) will denote the identity and zero matrices of \( \mathcal{M}_N(\mathbb{R}) \), and \( A \in \mathcal{M}_N(\mathbb{R}) \) a nonsingular matrix with nonzero diagonal entries. Related to \( A \) are the matrix \( D = D(A) \) whose nonzero entries are the diagonal entries of \( A \), a strictly lower triangular matrix \( L = L(A) \), and a strictly upper triangular matrix \( U = U(A) \) which are defined by

\[ A = D(I - L - U). \]  

(2.5)

In a project at Lockheed-California which modeled the flow of air over the surface of an aircraft, large full systems of equations of the form

\[ Ax = b \]  

(2.6)

arise, where \( b \in \mathbb{R}^N \) is a known \( N \)-vector, \( D = I \), and \( A \) is positive real (see [L]).

\(^3\) In addition, both of these concepts have generalizations to complex matrices, but our interest here is in real linear systems. Our usage of the term positive real agrees with the same terminology in [Y] and with what is called positive definite in [GV].

For an interesting discussion of why one must require \( z^* Az \) to be real and positive for all nonzero complex \( N \)-vectors in order to conclude that a real \( N \times N \) matrix is symmetric, see [Ha, Section 54, ff].
For $\omega \in \mathbb{R}$, define

$$L_{\omega} = L_{\omega}(A) = (I - \omega L)^{-1}[(1 - \omega)I + \omega U].$$

(2.7)

It is routine to verify that, for $\omega \neq 0$, (2.6) is equivalent to

$$x = L_{\omega} x + k$$

(2.8)

where $k = \omega(I - \omega L)^{-1}D^{-1}b$.

If we let

$$x^{(n+1)} = L_{\omega} x^{(n)} + k$$

(2.9)

then, as is shown in [Y, Section 3.3], $\{x^{(n)}\}$ will converge to the solution of (2.6) for any choice of an initial vector $x^{(0)}$ if and only if the spectral radius $\rho(L_{\omega})$ of $L_{\omega}$ is less than 1. If this latter holds, we say that $L_{\omega}$ converges.

This technique just described is called the method of successive over relaxation (SOR), and $L_{\omega}$ is called an SOR matrix. When $\omega = 1$, (2.9) is called the Gauss–Seidel method.

It is well known that if $L_{\omega}$ converges, then $0 < \omega < 2$, and the converse holds if $A$ is positive definite. The first assertion is due to V. Kahan, and the second to A. Ostrowski and E. Reich. See [Y, pp. 109, 113].

Since $A$ is positive definite if and only if it is positive real and has $(A - A^t) = 0$, it seems likely that a positive real matrix $A$ such that $(A - A^t)$ is "small" in some sense should have a convergent SOR matrix for a fairly large interval of $\omega$-values. That is the subject of this paper.

Convergence of linear iterative methods for positive real matrices has been investigated previously under various kinds of conditions; see [St, Theorem IV] and [O], where "near symmetry" is considered explicitly. See also [BP, Chapter 7].

We will use the following facts. When $A$ is positive real, its diagonal elements are positive, $\hat{A} = D^{-1/2}AD^{-1/2}$ is positive real, and $L_{\omega}(\hat{A}) = D^{1/2}L_{\omega}(A)D^{-1/2}$; see [Y, proof of Theorem 3.5], [GV, Section 10.1], or [St]. Thus, $L_{\omega}(A)$ and $L_{\omega}(\hat{A})$ have the same spectral radius, and $D(\hat{A}) = I$. Because of the latter fact, there is no loss of generality in assuming $D(A) = I$ when discussing $L_{\omega}(A)$, and we do this in what follows.

We make frequent use of the following facts established in [Y, Section 3.2].
(2.10) Let $A \in \mathcal{M}_N(\mathbb{R})$. $A$ is positive real if and only if $\frac{1}{2}(A + A^t)$ is positive definite. A symmetric matrix is positive definite if and only if all its eigenvalues are positive.

If $\lambda \in \mathbb{C}$, let $\text{Re}(\lambda)$ denote its real part and $\text{Im}(\lambda)$ its imaginary part. By (2.10) and a well-known theorem of Lyapunov given in [Y, Theorem 6.4], positive real implies $\text{Re}(\lambda) > 0$ for each eigenvalue $\lambda$ of $A$; but the converse is false. For example, by (2.10) the real matrix $\begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}$ is positive real if and only if $|s| < 2$, but for any $s \in \mathbb{R}$ and any eigenvalue $\lambda$ of $A$, $\text{Re}(\lambda) > 0$.

The following technical lemma is essential for our main results.

(2.11) **Lemma** Suppose $A = I - L - U$ is a positive real $N \times N$ matrix, $\omega$ is real and nonzero, $\lambda$ is an eigenvalue of $\mathcal{L}_\omega = \mathcal{L}_\omega(A)$, and $z$ is an eigenvector belonging to $\lambda$ such that $\|z\|_2 = 1$, which is chosen to be real if $\lambda$ is real. Let $\xi = -z^*\left[\frac{1}{2}(U^t + L)\right]z$ and $\kappa = -z^*\left[\frac{1}{2}(U^t - L)\right]z$, $a = \text{Re}(\xi)$, $b = \text{Im}(\xi)$, $c = \text{Re}(\kappa)$, $d = \text{Im}(\kappa)$, and $k = 1/\omega$. Then

(i) $\lambda = \frac{k - 1 - \xi - \kappa}{k + \xi - \kappa}$.

(ii) If $\mu$ is the smallest eigenvalue of $\frac{1}{2}(A + A^t)$, then $(2a + 1) \geq \mu > 0$ and $\mu \leq 1$.

(iii) $2|\xi| \leq \min\{\|U^t + L\|_\infty, \|U + L\|_\infty\} = M^+$ and $\max\{2|a|, 2|b|\} \leq \min\{M^+, \frac{1}{2}(A + A^t) - I\|_\infty\} = m^+$.

(iv) $2|\kappa| \leq \min\{\|U^t - L\|_\infty, \|U - L\|_\infty\} = M^-$ and $\max\{2|c|, 2|d|\} \leq \min\{M^-, \frac{1}{2}(A - A^t)\|_\infty\} = m^-.$

(v) $|\lambda| < 1$ if and only if $4bd < (2a + 1)(2k - 1 - 2c)$.

(vi) $\lambda$ is real if and only if $b = d = 0$.

(vii) If $bd \leq 0$ (in particular, if $\kappa = 0$ or $\lambda$ is real), then $|\lambda| < 1$ if $2c < (2 - \omega)/\omega$.

(viii) (Ostrowski–Reich) If also $A$ is symmetric (i.e., if $A$ is positive definite) then $|\lambda| < 1$ if and only if $0 < \omega < 2$.

**Proof** (i) $\mathcal{L}_\omega z = \lambda z$ implies $z^*[(1 - \omega)I + \omega U]z = \lambda z^*(I - \omega L)z$. We claim that $z^*[(1 - \omega)I + \omega U]z \neq 0$; for suppose this is zero. Then $1 = \omega \lambda z^* L z$ and $0 = z^*[(1 - \omega)I + \omega U]z = (1 - \omega) + \omega z^* U z$. Substituting for 1 in the latter and changing signs yields $\omega [I - z^* L z - z^* U z] = 0$, and $\omega \neq 0$ so it follows that
\( z^*[(I - L - U)]z = z^*Az = 0. \) However, letting \( z = x + iy \), where \( x, y \in \mathbb{R}^N \), the preceding implies \( 0 = \text{Re}(z^*Az) = x^tAx + y^tAy. \) Since \( A \) is positive real, this implies that \( x = y = 0 \), contrary to the fact that \( z \neq 0. \) It follows that \( z^*(I - \omega L)z \neq 0. \) Using this and \( k = 1/\omega \), we see that

\[
\lambda = \frac{z^*[((1 - \omega)I + \omega U)z]}{z^*(I - \omega L)z} = \frac{(1 - \omega) + \omega z^*[(\frac{1}{2}(U + L))z + \omega z^*[(\frac{1}{2}(U - L))z]}{1 - \omega z^*[(\frac{1}{2}(U^t + L))z + \omega z^*[(\frac{1}{2}(U^t - L))z]} = \frac{1 - \omega - \omega \xi - \omega \kappa}{1 + \omega \xi - \omega \kappa} = \frac{k - 1 - \xi - \kappa}{k + \xi - \kappa}.
\]

(ii) Note that \( 1 + 2\alpha = 1 + \xi + \bar{\xi} = z^*[I - \frac{1}{2}(U^t + L) - \frac{1}{2}(U + L)]z = z^*[(\frac{1}{2}(A + A^t)]z \geq \mu > 0. \) The inequalities follow from (2.10) and [ND, Theorem 11.7], since \( \frac{1}{2}(A + A^t) \) is positive definite and \( \|z\|_2 = 1. \) Also, \( N\mu \) cannot exceed the trace of \( \frac{1}{2}(A + A^t), \) which in this case is \( N. \) Hence, \( \mu \leq 1. \)

(iii) By (2.3),

\[
2|b| = |\xi - \bar{\xi}| = |z^*[(\frac{1}{2}(U^t + L) - \frac{1}{2}(U + L)]z| \\
\leq \|\frac{1}{2}(U^t + L) - \frac{1}{2}(U + L)\|_\infty \\
= \|\frac{1}{2}(U^t + L) + \frac{1}{2}(U + L)\|_\infty = \|\frac{1}{2}(A + A^t) - I\|_\infty.
\]

The sign change is valid by (2.1) since \( \frac{1}{2}(U^t + L) \) is strictly lower triangular and \( \frac{1}{2}(U + L) \) is strictly upper triangular.

Note also that \( 2|b| \leq 2|\xi| = |z^*(U^t + L)z| \leq \|U^t + L\|_\infty \) by (2.3). Similarly, \( 2|b| \leq 2|\bar{\xi}| \leq \|U + L\|_\infty. \) Thus, \( 2|\xi| \leq M^+ \) and \( 2|b| \leq m^+. \) In the same way, we get \( 2|a| \leq m^+. \)

(iv) By (2.3),

\[
2|d| = |\kappa - \bar{\kappa}| = |z^*[(\frac{1}{2}(U^t - L) - \frac{1}{2}(U - L)]z| \\
\leq \|\frac{1}{2}(U^t - L) - \frac{1}{2}(U - L)\|_\infty = \|\frac{1}{2}(A - A^t)\|_\infty.
\]

Also, \( 2|d| \leq 2|\kappa| = |z^*[(U^t - L)z| \leq \|(U^t - L)\|_\infty \) and similarly, \( 2|d| \leq 2|\kappa| \leq \|(U - L)\|_\infty. \) Thus, \( 2|\kappa| \leq M^- \) and \( 2|d| \leq m^- . \) The proof that \( 2|\kappa| \leq m^- \) is analogous.

(v) By elementary algebra and (i), it is evident that each of the following statements is equivalent to its successor: \( |\lambda| < 1; \)
\[ |k - 1 - \xi - \kappa|^2 < |k + \xi - \kappa|^2; (k - 1 - a - c)^2 + (b + d)^2 < (k + a - c)^2 + (b - d)^2; (b + d)^2 - (b - d)^2 < (k + a - c)^2 - (k - 1 - a - c)^2; 4bd < (2a + 1)(2k - 1 - 2c). \]

(vi) If \( \lambda \) is real, \( z \) is chosen to be a real eigenvector, whence both \( \xi \) and \( \kappa \) are real, so \( b = d = 0 \). Conversely, \( b = d = 0 \) says both \( \xi \) and \( \kappa \) are real, hence \( \lambda \) is real by (i).

(vii) \( 2c < (2 - \omega)/\omega \) implies \( 2k - 1 - 2c > 0 \) since \( k = 1/\omega \). By (ii), \( 2a + 1 > 0 \), so (vii) follows immediately from (v).

(viii) As noted after (2.9), \( \mathcal{L}_\omega \) convergent implies \( 0 < \omega < 2 \). Conversely, if \( 0 < \omega < 2 \) and \( A \) is symmetric, then \( U = L \), so \( \kappa = 0 = c = d \). Hence, \( bd = 0 \) and \( c = 0 < (2 - \omega)/\omega \). Thus, \( |\lambda| < 1 \) by (vii). This completes the proof of the Lemma.

As noted above, part (viii) of Lemma 2.11 is the well-known theorem of A. Ostrowski and E. Reich (see, e.g., [V, Section 3.4]) for real matrices. The proof given appears to be novel.

In all that follows, we will use the notation established in Lemma 2.11. In particular, for \( A \) in \( \mathcal{M}_p(\mathbb{R}) \), let

\[
M^+(A) = \min \{ \| U' + L \|_\infty, \| U + L \|_\infty \}, \tag{2.12}
\]

\[
m^+(A) = \min \{ M^+(A), \| \frac{1}{2}(A + A^t) - I \|_\infty \}, \tag{2.13}
\]

\[
M^-(A) = \min \{ \| U' - L \|_\infty, \| U - L \|_\infty \}, \tag{2.14}
\]

\[
m^-(A) = \min \{ M^-(A), \| \frac{1}{2}(A - A^t) \|_\infty \},
\]

\[
\mu(A) = \text{smallest eigenvalue of } \frac{1}{2}(A + A^t).
\]

Also, for an eigenvalue of \( \mathcal{L}_\omega(A) \) and associated eigenvector \( z, \xi, \kappa, a, b, c \) and \( d \) will have the special meanings defined in Lemma 2.11.

We are interested in finding conditions under which \( |\lambda| < 1 \), so parts (v) and (vii) of Lemma 2.11 are of special interest. By assuming that \( m^-(A) \) is small (i.e., the antisymmetric part of \( A \) is small), we can get \( 2c < (2 - \omega)/\omega \). Then Lemma 2.11(vii) gives \( |\lambda| < 1 \) if \( bd \leq 0 \). This leads us to look first for conditions that will yield \( bd \leq 0 \). Then in Section 3, we apply Lemma 2.11(v) by finding conditions which imply \( 4|bd| < (2a + 1)(2k + 1 - 2c) \).

Before proceeding, we give examples to show that any of the three norms in the definition of \( m^+ \) can exceed the other two, and the corresponding assertion about the norms in the definition of \( m^- \) is also true.
(2.15) Examples  We leave the algebraic details of the following examples as an exercise. Let \( p, q, r, s \in \mathbb{R} \), where \( p^2 + r^2 + s^2 < 1 \).

(i) Let

\[
A = \begin{bmatrix}
1 & p & q & 0 \\
p & 1 & r & s \\
-q & r & 1 & 0 \\
0 & s & 0 & 1
\end{bmatrix}
\]

Then

\[
\frac{1}{2}(A + A^t) = \begin{bmatrix}
1 & p & 0 & 0 \\
p & 1 & r & s \\
0 & r & 1 & 0 \\
0 & s & 0 & 1
\end{bmatrix}, \quad (U^t + L) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-2p & 0 & 0 & 0 \\
0 & -2r & 0 & 0 \\
0 & -2s & 0 & 0
\end{bmatrix},
\]

and

\[
(U + L^t) = \begin{bmatrix}
0 & -2p & 0 & 0 \\
0 & 0 & -2r & -2s \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

Thus, \( \frac{1}{2}(A + A^t) \) has characteristic polynomial \( (\lambda - 1)^2[(\lambda - 1)^2 - (p^2 + r^2 + s^2)] \) and \( \mu(A) = 1 - (p^2 + r^2 + s^2)^{1/2} \), so \( A \) is positive real by (2.10). Let \( \sigma_1 = \|\frac{1}{2}(A + A^t) - I\|_\infty \), \( \sigma_2 = \|U^t + L\|_\infty \), and \( \sigma_3 = \|U + L^t\|_\infty \). Then \( \sigma_1 = [p + |r| + |s|] \), \( \sigma_2 = \max\{2|p|, 2|r|, 2|s|\} \), and \( \sigma_3 = \max\{2|p|, 2(|r| + |s|)\} \). Thus \( \sigma_2 \leq \sigma_3 \), so \( m^+(A) = \min\{\sigma_1, \sigma_2\} \). If \( p = \frac{1}{2} \) and \( r = s = 0 \), then \( m^+ = \frac{1}{2} = \sigma_1 < \sigma_2 = \sigma_3 = 1 \), while if \( p = r = s = \frac{1}{3} \), then \( m^+ = \sigma_2 = 1 < \sigma_1 = 3/2 < \sigma_3 = 2 \).

Also, if \( A \) is replaced by \( A^t \) in this last example, then \( \sigma_2 \) and \( \sigma_3 \) exchange roles and we have \( m^+ = \sigma_3 = 1 < \sigma_1 = 3/2 < \sigma_2 = 2 \).

In summary, \( m^+ \) may be equal to any one of the \( \sigma_i \), while being unequal to the others.

(ii) Let

\[
B = \begin{bmatrix}
1 & p & q & 0 \\
-p & 1 & r & s \\
q & -r & 1 & 0 \\
0 & -s & 0 & 1
\end{bmatrix}
\]

where \( |q| < 1 \).
Then
\[
\frac{1}{2}(B + B') = \begin{bmatrix}
1 & 0 & q & 0 \\
0 & 1 & 0 & 0 \\
q & 0 & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}, \quad \text{and} \quad \frac{1}{2}(B - B') = \begin{bmatrix}
0 & p & 0 & 0 \\
-p & 0 & r & s \\
0 & -r & 0 & 0 \\
0 & -s & 0 & 0
\end{bmatrix},
\]
\[
(U^t - L) = \begin{bmatrix}
0 & 0 & 0 & 0 \\
-2p & 0 & 0 & 0 \\
0 & -2r & 0 & 0 \\
0 & -2s & 0 & 0
\end{bmatrix}, \quad \text{and} \quad (U - L) = \begin{bmatrix}
0 & -2p & 0 & 0 \\
0 & 0 & -2r & -2s \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{bmatrix}.
\]

The characteristic polynomial of \(\frac{1}{2}(B + B')\) is \((\lambda - 1)^2[(\lambda - 1)^2 - q^2]\), so \(\mu(B) = 1 - |q| > 0\), hence \(B\) is positive real. Let \(\tau_1 = \|\frac{1}{2}(B - B')\|_\infty\), \(\tau_2 = \|U^t - L\|_\infty\), and \(\tau_3 = \|U - L\|_\infty\). Then \(\tau_1 = |p| + |q| + |s|, \tau_2 = \max\{2|p|, 2|r|, 2|s|\}\), and \(\tau_3 = \max\{2|p|, 2|r| + |s|\}\). So we may proceed as in (i) above to produce examples of matrices \(B\) such that \(m^-(B)\) is any of the \(\tau_i\), while being unequal to the others.

Theorem 2.17 below, while rather special in character, has some interesting consequences. Lemma 2.16 contains technical details needed for (2.17).

(2.16) **Lemma** Suppose \(A = I - L - U\) is a positive real \(N \times N\) matrix, \(\lambda\) is an eigenvalue of \(L(A), \) and \(z\) is an eigenvector belonging to \(\lambda\) such that \(\|z\|_2 = 1\), which is chosen to be real if \(\lambda\) is real. Define \(\xi, \kappa, b, \) and \(d\) as in Lemma 2.11, and let \(\gamma = z^*(U - U^t)z\) and \(\delta = z^*(L - L^t)z\). Then

(i) \(4bd = \frac{1}{4}(|\gamma|^2 - |\delta|^2)\). Thus, \(bd \leq 0\) if and only if \(|\gamma| \leq |\delta|\).

(ii) If \(L = sU^t\) for some \(s \in \mathbb{R}\), then \(bd \leq 0\) if and only if \(\lambda\) is real or \(|s| \geq 1\).

**Proof** (i) Note first that \(2bi = \xi - \bar{\xi} = -z^*\left[\frac{1}{2}(U^t + L) - \frac{1}{2}(U + L)\right]z = \frac{1}{2}z^*[U^t(U - U^t) \cdots L - L^t]z = \frac{1}{4}(\gamma - \delta)\). Similarly, \(2di = \kappa - \bar{\kappa} = \frac{1}{2}(\gamma + \delta)\). Thus, \(4bd = -\frac{1}{4}(\gamma^2 - \delta^2)\). But \(\bar{\gamma} = -\gamma\) and \(\bar{\delta} = -\delta\), so \(\gamma^2 = -|\gamma|^2\) and \(\delta^2 = -|\delta|^2\). We conclude that \(4bd = \frac{1}{4}(|\gamma|^2 - |\delta|^2)\), which is nonpositive if and only if \(|\gamma| \leq |\delta|\).

(ii) Suppose \(L = sU^t\), so \(\delta = s\gamma\). If \(\lambda\) is real then \(bd = 0\) by Lemma 2.11(vi). If \(|s| \geq 1\), then \(|\gamma| \leq |\delta|\) and hence \(bd \leq 0\) by (i) above.
Conversely, suppose \(bd \leq 0\), in which case \(|\gamma| \leq |\delta| = |s||\gamma|\). Thus, \(|s| \geq 1\) or \(\gamma = \delta = 0\). If the latter is true, then, by the argument given in the proof of (i), \(\xi = \bar{\xi}\) and \(\kappa = \bar{\kappa}\). Hence, \(b = d = 0\), so \(\lambda\) is real by Lemma 2.11(vi).

(2.17) Theorem Suppose \(A = I - L - U\) is a positive real \(N \times N\) matrix; either \(L = sU^t\) for some real \(s\) such that \(|s| \geq 1\), or a dominant eigenvalue of \(\mathcal{L}_\omega(A)\) is real; and \(0 < \omega < 2/(1 + m^-)\). Then \(\mathcal{L}_\omega(A)\) converges.

Proof In either case, \(bd \leq 0\) (by Lemma 2.16 or Lemma 2.11(vi)). By Lemma 2.11(iv), \(2c \leq 2|c| \leq m^-\). Also, \(0 < \omega < 2/(1 + m^-)\) if and only if \(m^- < (2 - \omega)/\omega\), so \(\mathcal{L}_\omega(A)\) converges by Lemma 2.11(vii).

Observe that intuitively (2.16) and (2.17) say if \(L\) is “heavier” than \(U\) then \(\mathcal{L}_\omega(A)\) converges for certain \(\omega\).

Letting \(s = -1\) in (2.17) yields the following.

(2.18) Corollary If \(A - I\) is a skew-symmetric matrix, then \(\mathcal{L}_\omega(A)\) converges if \(0 < \omega < 2/(1 + m^-)\).

We need some notation for the next corollary. Suppose \(B = [b_{ij}]\) and \(C = [c_{ij}]\) are in \(\mathcal{M}_N(\mathbb{R})\). If \(b_{ij} \geq 0\) for all \(i, j\), we write \(B \geq 0\), and we write \(C \geq B\) if \(C - B \geq 0\). With respect to this partial ordering, \(\mathcal{M}_N(\mathbb{R})\) becomes a lattice-ordered ring with \(|B| = [b_{ij}]\). Moreover, as is shown in [BK, 8.1], for any \(B, C \in \mathcal{M}_N(\mathbb{R})\),

\[
|B + C| \leq |B| + |C| \quad \text{and} \quad |BC| \leq |B||C|. \tag{2.19}
\]

For \(A = I - L - U\), let \(C = C(A) = I - |L| - |U|\). In [BP], \(C(A)\) is called the comparison matrix of \(A\). Also, let \(C' = C'(A) = I - |L| + |U|\).

We need the following two facts from the Perron–Frobenius theory of nonnegative matrices as shown in [V, Theorem 2.7, 2.8].

(2.20) If \(B \geq 0\), the spectral radius \(\rho(B)\) is an eigenvalue of \(B\).

(2.21) If \(|B| \leq A\), then \(\rho(B) \leq \rho(A)\).

Note that \((I - X)^{-1} = I + X + \cdots + X^{N-1}\) when \(X\) is strictly lower triangular. So if \(0 < \omega \leq 1\), then

\[
\mathcal{L}_\omega(C(A)) = (I - \omega|L|)^{-1}[(I - \omega|I| + \omega|U|] \geq 0
\]

and hence has a dominant real eigenvalue by (2.20). Then, using the same expansion on \(\mathcal{L}_\omega(A)\), and (2.19) repeatedly, we get

(2.22) If \(0 < \omega \leq 1\), then \(|\mathcal{L}_\omega(A)| \leq \mathcal{L}_\omega(C(A))\).
Similarly, if $1 \leq \omega < 2$, then $-\mathcal{L}_\omega(C'(A)) \geq 0$, hence some dominant eigenvalue of $-\mathcal{L}_\omega(C')$ is real, and $|\mathcal{L}_\omega(A)| \leq -\mathcal{L}_\omega(C')$.

(2.23) **Corollary** Let $A = I - L - U$, $C = I - |L| - |U|$, and $C' = I - |L| + |U|$.

(i) If $C$ is positive real, $0 < \omega \leq 1$, and $\omega < 2/[1 + m^-(C)]$ then $\mathcal{L}_\omega(A)$ converges.

(ii) If $C'$ is positive real, $1 \leq \omega < 2$ and $\omega < 2/[1 + m^+(C')]$ then $\mathcal{L}_\omega(A)$ converges.

**Proof** (i) Since $C$ is positive real and $\mathcal{L}_\omega(C) \geq 0$ if $0 < \omega \leq 1$, (2.20) enables us to apply Theorem 2.17 and hence to conclude that $\rho(\mathcal{L}_\omega(C)) < 1$. Thus, by (2.21) and (2.22), $\rho(\mathcal{L}_\omega(A)) < 1$.

(ii) Essentially the same reasoning as in the proof (i) yields $\rho(\mathcal{L}_\omega(A)) \leq \rho(\mathcal{L}_\omega(C')) < 1$ if $\omega < 2/[1 + m^-(C')]$. After noting that $C' = I - |L| - (-|U|)$, it is routine to verify that $m^{-}(C') = m^{+}(C)$.

The previous result is analogous to Theorem 5.18 of Chapter 7 of [BP], which implies that if $A$ is diagonally dominant, then $\mathcal{L}_\omega(A)$ converges if $0 < \omega < 2/[1 + \rho(|L| + |U|)]$.

### 3. ADDITIONAL SUFFICIENT CONDITIONS FOR THE CONVERGENCE OF SOR

In this section, we present some conditions that guarantee convergence of $\mathcal{L}_\omega(A)$ for positive real $A$, even if $bd > 0$ and there is no dominant real eigenvalue for $\mathcal{L}_\omega(A)$. We begin with an example which shows that this can, indeed, be the case.

(3.1) **Example** Let $A = \begin{bmatrix} 1 & 1 \\ s & 1 \end{bmatrix}$, where $s \in \mathbb{R}$ and $0 < s < 1$. Apply (2.10) to see that $A$ is positive real. By Lemma 2.16(b), since $|s| < 1$, $bd > 0$. A routine calculation shows that the characteristic polynomial of $\mathcal{L}_\omega(A)$ is

$$|\lambda I - \mathcal{L}_\omega| = |\lambda(I - \omega L) - (1 - \omega)I - \omega U|$$

$$= \lambda^2 - [2(1 - \omega) + s\omega^2]\lambda + (1 - \omega)^2.$$ 

So the eigenvalues of $A$ will fail to be real precisely when $[2(1 - \omega) + s\omega^2]^2 - 4(1 - \omega)^2 < 0$; equivalently if

$$\omega^2 s [4(1 - \omega) + \omega^2 s] < 0.$$ 

Since $0 < s < 1$, it follows that the
eigenvalues of $L_{\omega}(A)$ are nonreal if and only if

$$\frac{2}{1 + \sqrt{1 - s}} < \omega < \frac{2}{1 - \sqrt{1 - s}}.$$  

When the eigenvalues of $L_{\omega}$ are real, applying Theorem 2.17 yields that $L_{\omega}$ converges when

$$0 < \omega < \frac{2}{1 + m^{-}(A)} = \frac{2}{1 + \frac{1}{1 - s} - \frac{3}{2} - s}.$$  

A routine calculation shows that

$$\frac{2}{1 + \sqrt{1 - s}} < \frac{4}{3 - s} \quad \text{if} \quad 0 < s < 1$$  

and we may conclude that $L_{\omega}(A)$ converges at least for

$$0 < \omega \leq \frac{2}{1 + \sqrt{1 - s}}.$$  

By the above $L_{\omega}(A)$ has a dominant nonreal eigenvalue when

$$\frac{2}{1 + \sqrt{1 - s}} < \omega < 2.$$  

If $\lambda_0$ is that eigenvalue, $|\lambda_0|^2 = \lambda_0 \bar{\lambda}_0 = (1 - \omega)^2$, the constant term of the characteristic polynomial of $L_{\omega}(A)$. Since $(1 - \omega) < 1$ for $0 < \omega < 2$, we conclude in this case that $L_{\omega}(A)$ actually converges for $0 < \omega < 2$.

For this small matrix, this same conclusion may be reached easily without using Theorem 2.17, but the presence of a dominant complex eigenvalue causes much more difficulty for larger matrices.

We want conditions that imply $4bd < (2a + 1)(2k - 1 - 2c)$ when $bd > 0$. This necessarily involves using a lower bound for $(2a + 1)$. By 2.11(ii), the greatest lower bound we know for $(2a + 1)$ is the minimal eigenvalue $\mu$ of $\frac{1}{2}(A + A^T)$.

(3.2) Theorem Suppose $A$ is positive real. Then $L_{\omega}(A)$ converges if

$$0 < \omega < \frac{2}{1 + m^{-}[1 + (m^+/\omega)]}.$$
Proof By Lemma 2.11(iii) and (iv), $4bd \leq m^+ m^-$ and by (ii) and (iv), $(2k - 1 - m^-) \leq (2a + 1)(2k - 1 - 2c)$. So Lemma 2.11(v) implies that $L_\omega(A)$ converges if $m^+ m^- < \mu(2k - 1 - m^-)$. Since $k = 1/\omega > 0$, simple algebraic manipulation yields (3.2).

Theorem 3.4 below is better than Theorem 3.2 when $M^- = m^- \cdots$ — i.e. it yields a larger interval of values of $\omega$ for which $L_\omega(A)$ converges. The following lemma will be useful in proving (3.4).

(3.3) **Lemma** Suppose $\rho \in \mathbb{C}$, $a = \text{Re}(\rho)$, $b = \text{Im}(\rho)$, $k \in \mathbb{R}$, $2a + 1 > 0$, $2k - 1 > 0$, and $|(\xi - 1 - \rho)/(k + \xi)| < 1$. If $\kappa \in \mathbb{C}$ and

$$|\kappa| < \frac{(2a + 1)(2k - 1)}{2\sqrt{(2a + 1)^2 + 4b^2}}$$

then

$$\left| \frac{k - 1 - \xi - \kappa}{k + \xi - \kappa} \right| < 1.$$

Proof Let $\mathbb{C}' = \mathbb{C} \setminus \{k + \xi\}$, and define $f: \mathbb{C}' \to \mathbb{R}$ by letting $f(\kappa) = \frac{(k - 1 - \xi - \kappa)/(k + \xi - \kappa)}$. Clearly $f$ is continuous and $\mathbb{C}'$ is connected. By assumption, $f(0) < 1$ and $Z = f^{-1}(1)$ is a nonempty closed subset of $\mathbb{C}'$ that does not contain 0. So there is an $\kappa_0 \in Z$ whose (positive) distance to 0 is minimal.

Let $S = \{\kappa \in \mathbb{C} : |\kappa| < |\kappa_0|\}$, so $1 \notin f(S)$. Since $f$ is continuous and $S$ is connected, $f(S)$ is connected; and $0 \in S$. Since $1 \notin f(S)$ and $f(0) < 1$ by hypothesis, we have $f(\kappa) < 1$ for all $\kappa \in S$; i.e. if $|\kappa| < |\kappa_0|$ then $f(\kappa) < 1$.

We want to show that $|\kappa_0|$ is the expression given in the Lemma, so we need to calculate $\kappa_0$. Observe that $\kappa_0 = (c_o, d_o)$ is the point where $h(c, d) = c^2 + d^2$ has its minimum along the curve $\{(c, d) : \kappa = c + id$ and $|k - 1 - \xi - \kappa|^2 = |k + \xi - \kappa|^2\}$. As in the proof of Lemma 2.11(v), one can show that $|k - 1 - \xi - \kappa|^2 = |k + \xi - \kappa|^2$ is equivalent to $(2a + 1)(2k - 1 - 2c) + 4bd = 0$.

Thus, if one defines $g(c, d) = (2a + 1)(2k - 1 - 2c) + 4bd$, then $\kappa_0 = (c_o, d_o)$ is the point where $h(c, d)$ has its minimum along $g(c, d) = 0$. This point can be found using the Lagrange Multiplier Theorem (see [A, 9.14]) since $h$ and $g$ are smooth functions. That theorem says that for some $t \in \mathbb{R},$

$$t \left( \frac{\partial h}{\partial c} (c_o), \frac{\partial h}{\partial d} (d_o) \right) = \left( \frac{\partial g}{\partial c} (c_o), \frac{\partial g}{\partial d} (d_o) \right);$$

that is, $t(2c_o, 2d_o) = (-2(2a + 1), -4b)$. We have $2a + 1 > 0$ by assumption, $t \neq 0$ and $c_o \neq 0$; so $d_o/c_o = 2b/(2a + 1)$ or $d_o = 2bc_o/(2a + 1)$. Hence, $0 = g(c_o, d_o) = (2a + 1)(2k - 1 - 2c_o) -$
$8b^2c_0/(2a + 1)$. Solving for $c_0$ yields

$$c_0 = \frac{4b}{2[(2a + 1)^2 + 4b^2]}.$$ 

Thus,

$$|\kappa_0|^2 = c_0^2 + d_0^2 = c_0^2\left[1 + \frac{4b^2}{(2a + 1)^2}\right] = \frac{(2a + 1)^2(2k - 1)^2}{4[(2a + 1)^2 + 4b^2]}.$$ 

Since both $(2a + 1)$ and $(2k - 1)$ are positive,

$$|\kappa_0| = \frac{(2a + 1)(2k - 1)}{2\sqrt{(2a + 1)^2 + 4b^2}}.$$ 

(3.4) **Theorem.** If $A = I - L - U$ is a positive real $N \times N$ matrix, and

$$0 < \omega < \frac{2}{1 + M^{-}\sqrt{1 + (m^+/\mu)^2}},$$

then $L_\omega(A)$ converges.

**Proof.** Observe first that

$$\omega < \frac{2}{1 + M^{-}\sqrt{1 + (m^+/\mu)^2}}$$

is equivalent to

$$M^{-} < \frac{(2k - 1)\mu}{\sqrt{\mu^2 + (m^+)^2}}.$$ 

Next, if $\lambda$ is an eigenvalue of $L_\omega(A)$, then by Lemma 2.11(i), $|\lambda| = (k - 1 - \xi - \kappa)/(k + \xi - \kappa)$, and by Lemma 2.11(iv) and the above,

$$|\kappa| \leq M^{-} < \frac{(2k - 1)\mu}{\sqrt{\mu^2 + (m^+)^2}}.$$ 

Also, $2|b| \leq m^+$ by Lemma 2.11(iii), so

$$|\kappa| < \frac{(2k - 1)\mu}{\sqrt{\mu^2 + 4b^2}}.$$ 

Finally, the function $h: \mathbb{R} \to \mathbb{R}$ given by $h(x) = x/\sqrt{x^2 + p^2}$ is strictly increasing for $p > 0$. Hence, $\mu \leq 2a + 1$ implies

$$|\kappa| < \frac{2k - 1}{2} \frac{2a + 1}{\sqrt{(2a + 1)^2 + 4b^2}}.$$
By the proof of Lemma 2.2(v), $|k - 1 - \xi|/|k + \xi| < 1$ is equivalent to $0 < (2a + 1)(2k - 1)$, which is true by hypothesis and Lemma 2.11(i). So Lemma 3.3 applies, hence $|\xi| < 1$.

It is easy to verify that
\[
\frac{2}{1 + M^{-}(1 + m^{+}/\mu))} \leq \frac{2}{1 + M^{-}\sqrt{1 + (m^{+}/\mu)^2}},
\]
hence Theorem 3.4 yields a larger interval of $\omega$-values than Theorem 3.2, at least when $M^{-} = m^{-}$.

Theorem 3.4 and its proof were inspired by [O, Theorem IV]. An immediate comparison of the two theorems is not possible since Ostrowski used the norm now commonly denoted by $\|A\|_2 = \sup\{\|Ax\|_2 : \|x\|_2 = 1\}$ and he considered only the Gauss–Seidel matrix $L_{1}(A) = (I - L)^{-1}U$.

Also, he allowed any splitting of $A, A = M + T$, such that $M$ is positive definite and $T$ has zero diagonal, and he defined his constants $\Lambda_1$ and $\Lambda_2$ using max instead of min. Nevertheless, if we use the particular splitting $A = \frac{1}{2}(A + A^t) + \frac{1}{2}(A - A^t)$, then $\Lambda_1$ and $\Lambda_2$ are analogous to $\frac{1}{2}M^+$ and $\frac{1}{2}M^{-}$; his $\lambda_0$ is the same as our $\mu$; and little alteration is needed in his proof of Theorem IV, except making use of (2.3) as was done in the proof of Lemma 2.11(iii) and (iv), to show:

\[(3.5) \text{Theorem (Ostrowski)} \quad \text{If } A \text{ is positive real, then } L_{1}(A) \text{ converges if } M^{-} < \mu/\sqrt{\mu^2 + (\frac{1}{2}M^{+})^2} - \frac{1}{2}M^{+}.
\]

To compare (3.5) with our Theorem 3.4, let $\omega = 1$ in (3.4) to get, after some simple algebra, that $L_{1}(A)$ converges if $M^{-} < \mu/\sqrt{\mu^2 + (m^{+})^2}$.

Since $M^{-}$ is a measure of the nonsymmetry of $A$, we will say that (3.4) is stronger than (3.5) if it yields convergence of $L_{1}(A)$ for possibly larger values of $M^{-}$, that is if
\[
\sqrt{\mu^2 + (\frac{1}{2}M^{+})^2} - \frac{1}{2}M^{+} < \mu/\sqrt{\mu^2 + (m^{+})^2}. \quad (3.6)
\]

To verify (3.6), observe first that it suffices to prove the inequality with $M^{+}$ instead of $m^{+}$ on the right side, since $m^{+} \leq M^{+}$. Make that change, rationalize the left side and simplify to get
\[
2\mu/\sqrt{\mu^2 + (M^{+})^2} < \sqrt{4\mu^2 + (M^{+})^2} + M^{+}.
\]
Square and simplify to get
\[
2\mu^2 (\mu^2 + (M^+)^2) < 2\mu^2 + (M^+)^2 + M^+ \sqrt{4\mu^2 + (M^+)^2}.
\] (3.7)

Finally, \(0 < \mu \leq 1\) by Lemma 2.11(ii), so the left side of (3.7) is less than or equal to \(2\mu^2 + (M^+)^2\), which is clearly less than the right side of (3.7).

We close with a few remarks and examples. Corollary 2.18 is an immediate consequence of Theorem 3.4 since if \(A - I\) is skew-symmetric, then \(m^+ = 0\).

The bounds provided by Theorems 2.17 and 3.4 are rather conservative. For example, if \(A = \begin{bmatrix} 1 & s \\ 0 & 1 \end{bmatrix}\), then by Theorem 2.17, \(\mathcal{L}_\omega(A)\) converges for \(0 < \omega < 2/(1 + |s|)\) provided that \(|s| < 2\). But it is clear that \(\mathcal{L}_\omega(A) = \begin{bmatrix} 1 - \omega & -\omega s \\ 0 & 1 - \omega \end{bmatrix}\) converges for any \(s\) and any \(\omega\) between 0 and 2. The conservativeness of the bounds is also very apparent for \(A_1\) and \(A_2\) below, where \(m^-\) is large. However, they are not bad predictors for \(A_3\) and \(A_4\), which are "nearly diagonally dominant" and "nearly symmetric".

The interval of convergence for \(\mathcal{L}_\omega(A)\) provided by Theorem 3.4 is strictly smaller than the one provided by Theorem 2.17 unless \(m^+ = 0\) or \(m^- = 0\). However, all the positive real examples we have found have \(\mathcal{L}_\omega\) convergent for all \(\omega\) in the larger interval of Theorem 2.17, whether or not they satisfy the hypotheses of (2.17).

\[(3.8)\) Examples To simplify notations, for a matrix \(A_i\) let \(m_i^+ = m^+(A_i), m_i^- = m^-(A_i), M_i^- = M^-(A_i),\) and \(M_i^+ = M^+(A_i + A_i^t)\). Also let

\[
\alpha_i = \frac{2}{1 + m_i^-}, \quad \beta_i = \frac{2}{1 + m_i^- (1 + m_i^+/\mu)}, \quad \text{and} \quad \gamma_i = \frac{2}{1 + M_i - \sqrt{1 + (m_i^+/\mu)^2}}.
\]

These correspond to the intervals for \(\omega\) guaranteed by (2.17), (3.2), and (3.4). Finally, let \(\rho_i(\omega)\) denote the spectral radius of \(\mathcal{L}_\omega(A_i)\). The table below contrasts the predictive power of Theorems 2.17, 3.2, and 3.4 by displaying spectral radii of \(\mathcal{L}_\omega(A)\) for selected values of \(\omega\). The
eigenvalues for the table were calculated by the EIGRF routine in the IMSL software package \([\Pi]\), and rounded to three significant figures. The table is for these matrices:

\[
A_1 = \begin{bmatrix}
1 & .5 & .3 & .2 \\
.7 & 1 & .7 & -.4 \\
.5 & .7 & 1 & -.2 \\
1 & 0 & 0 & 1
\end{bmatrix}
\quad A_2 = \begin{bmatrix}
1 & -.5 & -.3 & -.2 \\
.7 & 1 & -.7 & .4 \\
.8 & .7 & 1 & .5 \\
1 & 0 & 0 & 1
\end{bmatrix}
\]

\[
A_3 = \begin{bmatrix}
1 & -.5 & .3 & .2 \\
-5 & 1 & .2 & .3 \\
.2 & .3 & 1 & .5 \\
.3 & .2 & .5 & 1
\end{bmatrix}
\quad A_4 = \begin{bmatrix}
1 & .5 & .2 & .3 \\
.5 & 1 & .2 & .3 \\
.2 & .3 & 1 & .5 \\
.3 & .2 & .5 & 1
\end{bmatrix}
\]

Finally, here is an example for which all the relevant eigenvalues can be easily calculated explicitly.

Let

\[
B = \begin{bmatrix}
1 & p & q & 0 \\
-p & 1 & p & 0 \\
q & -p & 1 & 0 \\
0 & 0 & 0 & 1
\end{bmatrix}
\]

where \(0 < p < 1\) and \(0 < q < 1\).

By the calculation made in Example 2.15, it is easy to see that

\[
\alpha(B) = \frac{2}{1 + 2p},
\]

\[
\beta(B) = \frac{2}{1 + 2p \left( 1 + \frac{q}{1 - q} \right)},
\]

and

\[
\gamma(B) = \frac{2}{1 + 2p \sqrt{1 + \left( \frac{q}{1 - q} \right)^2}}.
\]

\(B\) is "nearly symmetric" if \(p\) is small, and \(\beta(B)\) is smaller than \(\gamma(B)\), so the result of Theorem 3.4 improves on the result of (3.2).
<table>
<thead>
<tr>
<th>Matrix</th>
<th>$\mu_i$</th>
<th>$\alpha_i$</th>
<th>$\beta_i$</th>
<th>$\gamma_i$</th>
<th>$\rho_i(0.2)$</th>
<th>$\rho_i(0.4)$</th>
<th>$\rho_i(0.6)$</th>
<th>$\rho_i(0.8)$</th>
<th>$\rho_i(1)$</th>
<th>$\rho_i(1.2)$</th>
<th>$\rho_i(1.4)$</th>
<th>$\rho_i(1.6)$</th>
<th>$\rho_i(1.8)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$A_1$</td>
<td>0.040</td>
<td>1.18</td>
<td>0.067</td>
<td>0.041</td>
<td>0.978</td>
<td>0.952</td>
<td>0.920</td>
<td>0.878</td>
<td>0.821</td>
<td>0.732</td>
<td>0.497*</td>
<td>0.614*</td>
<td>0.855*</td>
</tr>
<tr>
<td>$A_2$</td>
<td>0.585</td>
<td>0.727</td>
<td>0.378</td>
<td>0.370</td>
<td>0.844</td>
<td>0.703</td>
<td>0.571*</td>
<td>0.441</td>
<td>0.750*</td>
<td>1.40</td>
<td>2.12</td>
<td>2.75</td>
<td>3.13*</td>
</tr>
<tr>
<td>$A_3$</td>
<td>0.293</td>
<td>1.82</td>
<td>1.39</td>
<td>1.17</td>
<td>0.937</td>
<td>0.862</td>
<td>0.771</td>
<td>0.651</td>
<td>0.454</td>
<td>0.330*</td>
<td>0.469*</td>
<td>0.645*</td>
<td>0.823*</td>
</tr>
<tr>
<td>$A_4$</td>
<td>0.448</td>
<td>1.82</td>
<td>1.50</td>
<td>1.59</td>
<td>0.895*</td>
<td>0.777*</td>
<td>0.640*</td>
<td>0.468*</td>
<td>0.290</td>
<td>0.341*</td>
<td>0.509*</td>
<td>0.674*</td>
<td>0.829*</td>
</tr>
</tbody>
</table>

* Denotes no dominant real eigenvalues for $L_\omega(A_i)$. 
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