NEARLY PSEUDOCOMPACT EXTENSIONS

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Abstract. A Tichonov space $X$ is nearly pseudocompact if $\nu X - X$ is dense in $\beta X - X$. This paper studies the relations between a space $X$ and its canonical nearly pseudocompact extension $\delta X$. We find a new characterization of nearly pseudocompact spaces, and a characterization of those $X$ for which $\delta X$ is pseudocompact. We also find conditions on $X$ which make $\delta X$ a $k$-space, a $k'$-space, and a $K$-space, and consider the images and pre-images of such $X$ under pseudo-open maps.

1. Introduction. This paper is a continuation of [HR], and terminology not otherwise defined here may be found in that paper or in [GJ]. All spaces considered will be Tichonov unless explicitly stated otherwise. As usual, $\beta X$ will denote the Stone-$\check{C}$ech compactification and $\nu X$ the Hewitt realcompactification of a space $X$.

Noting that a space $X$ is pseudocompact precisely when $\nu X - X = \beta X - X$, we call $X$ nearly pseudocompact when $\nu X - X$ is dense in $\beta X - X$. For any space $X$, we define $\delta X = \beta X - (K - X)$, where $K$ (or $K_X$, if needed) = cl$_{\beta X}$ $(\nu X - X)$. Since $X$ is nearly pseudocompact if and only if $X = \delta X$, and $\delta \delta X = \delta X$, we refer to $\delta X$ as the (canonical) nearly pseudocompact extension of $X$. The hard sets of $X$ form a key tool in the study of $\delta X$; a subset $H$ of $X$ is called a hard set of $X$ if $H$ is closed in $X \cup K$.

Section 2 of this paper presents a necessary and sufficient condition on $X$ for $\delta X$ to be pseudocompact. Section 3 investigates the relations between the behaviour of $\delta X$ with respect to its compact sets and the behaviour of $X$ with respect to its hard sets. Section 4 considers mapping theorems on such spaces under pseudo-open maps, while Section 5 presents some examples.

2. When is $\delta X$ pseudocompact?

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Verifications of the following facts about hard sets and nearly pseudocompact spaces may be found in [R 1], [R 2], or [HR]. For any space $X$, let $H(X)$ denote the family of hard subsets of $X$.

2. 1.

a ) $H \in H(X)$ if and only if $H = B \cap X$ for some compact $B \subseteq \delta X$. Moreover every hard set of $X$ is realcompact.

b ) $\delta X = \cup \{ cl_{\delta X}(H) : H \in H(X) \}$.

c ) If $H \subseteq X$ is compact, or a closed subset of a hard subset of $X$, or a finite union of hard subsets of $X$, then $H \in H(X)$.

2. 2. Definition. If $p$ is a topological property, let $N_p(X)$ be the set of all $x$ in $X$ such that $x$ fails to have a closed neighborhood with property $p$. If $N_p(X) = \emptyset$, then $X$ is called a locally-$p$ space, while if $X - N_p(X)$ is dense in $X$, then $X$ is called an almost locally-$p$ space. If $N_p(X) = X$, then $X$ is said to be a nowhere locally-$p$ space.

Note that $N_p(X)$ is always a closed subset of $X$. We abbreviate compactness by $p = k$, hardness by $p = h$ and realcompactness by $p = rc$. The set $N_k(X)$ of points at which $X$ is not locally compact is traditionally denoted by $R(X)$ and is called the residue of $X$.

2. 3.

a ) Every pseudocompact space and every nowhere locally realcompact space is nearly pseudocompact.

b ) Regular closed subspaces and finite unions of (nearly) pseudocompact spaces are (nearly) pseudocompact.

c ) $N_k(\delta X) = K_X \cap X = N_{rc}(X)$.

d ) $X = \delta X$ if and only if $X = X_1 \cup X_2$, where $X_1$ is a regular closed, almost locally compact subspace that is pseudocompact, $X_2$ is a regular closed, nowhere locally realcompact subspace, and $\text{int}_X (X_1 \cap X_2) = \emptyset$. (Indeed,
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\[ X_1 = \text{cl}_{\delta X} \left[ \delta X - N_k(\delta X) \right] \text{ and } X_2 = \text{cl}_{\delta X} \left[ \delta X - X_1 \right]. \]

In the characterization of \( \delta X \) in 2.3 d, let us observe that \( X_2 = \text{cl}_{\delta X} \left[ \delta X - X_1 \right] \) is always a subset of \( X \):

2.4. Lemma. \( X_2 = \text{cl}_{\delta X} \left[ K \cap X \right] = \text{cl}_{\delta X} \left[ \text{int}_X N_{rc}(X) \right]. \)

Proof. By 2.3 c), \( \delta X - X_1 = \delta X - \text{cl}_{\delta X} \left[ \delta X - N_k(\delta X) \right] = \text{int}_{\delta X} N_k(\delta X) = \text{int}_{\delta X} (K \cap X) \). Thus \( X_2 = \text{cl}_{\delta X} \left[ \text{int}_{\delta X} (K \cap X) \right] \). Since \( K \cap X \subseteq X \), we know that \( \text{int}_{\delta X} (K \cap X) \subseteq \text{int}_X (K \cap X) \). Suppose \( G \) is open in \( \delta X \) and \( G \cap X = \text{int}_X (K \cap X) \). Since \( X \) is dense in \( \delta X \), both \( G \) and \( G \cap X \) have the same closure in \( \delta X \). By 2.3 c), \( K \cap X = \text{int}_X N_k(\delta X) \) is closed in \( \delta X \), so \( \text{cl}_{\delta X} G \subseteq K \cap X \). Hence \( G \subseteq X \) and \( \text{int}_X (K \cap X) \) is open in \( \delta X \). Thus \( \text{int}_{\delta X} (K \cap X) = G \), and \( \text{cl}_{\delta X} \text{int}_{\delta X} (K \cap X) = \text{cl}_{\delta X} G \subseteq X \) implies \( X_2 = \text{cl}_{\delta X} \left[ \text{int}_X N_{rc}(X) \right] \) by 2.3 c).

From this, it follows quickly that \( X_1 = \text{cl}_{\delta X} \left[ X - N_{rc}(X) \right] \).

We use these facts to provide a new characterization of nearly pseudocompact spaces and a characterization of those spaces for which \( \delta X \) is pseudocompact.

2.5. Theorem. \( X \) is nearly pseudocompact if and only if \( X^{(1)} \) is nearly pseudocompact.

Proof. If \( X = \delta X \), then the regular closed subspace \( X^{(1)} \) of \( X \) is nearly pseudocompact by 2.3 b). By 2.3 a) and 2.4, \( X_2 \) is a nearly pseudocompact subspace of \( X \), and it is clear that \( X = X^{(1)} \cup X_2 \). So, if \( X^{(1)} \) is nearly pseudocompact, so is \( X \) by 2.3 b).

2.6. Theorem. \( \delta X \) is pseudocompact if and only if \( X_2 = \text{cl}_{\delta X} \text{int}_X N_{rc}(X) \) is pseudocompact.

Proof. Suppose \( \delta X \) is pseudocompact. Then \( X_2 \), being a regular closed subset of \( \delta X \) by 2.3 d) and 2.4, is pseudocompact by 2.3 b). If \( X_2 \) is pseudocompact, then since \( X_1 \) is pseudocompact by 2.3 d), so is \( \delta X \) by 2.3 b).
2.7. Corollary. If $N_k(X)$ or $N_{rc}(X)$ is pseudocompact, then so is $\delta X$.

Proof. In either case, $cl_X int_X N_{rc}(X)$ is a regular closed subset of a pseudocompact space, and hence is pseudocompact by 2.3 b). Hence $\delta X$ is pseudocompact by 2.6.

In [R2], it is asked if there is an internal (to $X$) characterization of the property $vX-X$ is closed in $\beta X-X$. We can now give some necessary conditions for this to happen. As shown in Example 5.4, however, these conditions are not sufficient.

2.8. Theorem. Suppose $vX-X$ is closed in $\beta X-X$. Then

a) $cl_X int_X N_{rc}(X)$ is pseudocompact, and

b) every $C$-embedded realcompact subset of $X$ is hard in $X$.

Proof. a) If $vX-X$ is closed in $\beta X-X$, then $K-X=cl_{\beta X}(vX-X) \cap (\beta X-X)=vX-X \subseteq v(\delta X)-\delta X \subseteq \beta (\delta X)-\delta X=K-X$. So $v(\delta X)-\delta X=\beta (\delta X)-X$, whence $\delta X$ is pseudocompact. So a) holds by 2.6.

b) Suppose $B$ is a realcompact $C$-embedded subset of $X$. Since $vX=X \cup K$, $cl_X \cup K(B)=cl_{vX}(B)=vB=B$ by [GJ, 8.10 (a)]. Thus $B$ is a hard set of $X$.

Recall from [R2] that a space $X$ is called $h$-normal if $X \cup K$ is normal. Thus every hard subset of an $h$-normal space is $C$-embedded since it is closed in $X \cup K$ by definition. Hence we have:

2.9. Corollary. If $X$ is $h$-normal and $vX-X$ is closed in $\beta X-X$, then the hard subsets of $X$ are precisely the closed realcompact subsets of $X$.

We closed this section with a characterization of hard sets among the closed, $C^*$-embedded subsets of $X$.

2.10. Theorem. Suppose $B$ is a $C^*$-embedded, closed subset of $X$. Then $B$ is hard if and only if $B$ is realcompact and $\delta B \subseteq \delta X$.
Proof. If $B$ is realcompact, then $\delta B = \beta B = cl_{\beta X} B$, since $B$ is $C^*$-embedded in $X$. Thus $\delta B \subseteq \delta X$ implies $cl_{\beta X} B \subseteq \delta X$, whence $B$ is hard. The other direction is clear.

3. Relations between $X$ and $\delta X$.

It was shown in [R1] that a space $X$ is locally realcompact if and only if $X$ is locally hard, i.e. $N_{rc}(X) = \phi \iff N_h(X) = \phi$. We can do better:

3.1. Theorem. For any $X$, $N_{rc}(X) = N_h(X)$.

Proof. Since every hard set is realcompact by 2.1 a), $N_{rc}(X) \subseteq N_h(X)$. Since $N_{rc}(X) = X \cap K$, if $x \in X - N_{rc}(X)$, then by regularity, there is an $X \cup K$ open neighborhood $W$ of $x$ such that $cl_{X \cup K}(W) \subseteq X - N_{rc}(X)$. Thus $cl_X(W) = cl_{X \cup K}(W)$ is hard in $X$, and $x \in X - N_h(X)$.

3.2. Definition. Suppose $p$ is a topological property such that if $f$ is a homeomorphism from $X$ onto $Z$ and $S$ is a subset of $X$ with property $p$, then $f[S]$ has $p$ as well. In addition, we shall assume that property $p$ is possessed by all compact sets. Let $\mathcal{F}_p$ be the cover of $X$ by all closed subsets having property $p$. For this definition, $X$ need not be Tichonov.

a) A space $X$ is called a $p$-space if whenever $A \subseteq X$ meets each element of $\mathcal{F}_p$ in a closed set, then $A$ is closed in $X$.

b) A space $X$ is called a $p^*$-space if whenever each real-valued function $f$ on $X$ is continuous on each member of $\mathcal{F}_p$, then $f \in C(X)$.

c) A point $x$ is a $p^*$-point of $X$ if for every $A \subseteq X$ with $x \in cl(A)$, then $x \in cl(A \cap F)$ for some $F \in \mathcal{F}_p$. Let $S_p(X)$ be the set of non $p^*$-points of $X$. If $S_p(X) = \phi$, then $X$ is called a $p^*$-space.

In case $p$ is compactness, see [A] and [N].

3.3. Lemma. For any property $p$, if $A \subseteq X$ is such that $A \cap F$ is closed for every $F \in \mathcal{F}_p$, then $cl(A) - A \subseteq S_p(X)$.
Proof. Let \( x \in \text{cl}(A) \) and suppose \( x \notin S_p(X) \). Then there is a set \( F \in \mathcal{F}_p \) with \( x \in \text{cl}(A \cap F) = A \cap F \) by hypothesis. Thus \( x \in A \).

From this, it follows that

3.4. Theorem. For any property \( p \), every \( p' \)-space is a \( p \)-space.

3.5. Lemma. For any \( X \), \( S_{rc}(X) \subseteq S_h(X) \subseteq S_k(\delta X) \subseteq N_{rc}(X) \).

Proof. That \( S_{rc}(X) \subseteq S_h(X) \) is equivalent to saying every \( h' \)-point of \( X \) is an \( rc' \)-point of \( X \). This is immediate from 2.1 a), since every hard set is closed and realcompact.

That \( S_h(X) \subseteq S_k(\delta X) \) is the same as saying every \( k' \)-point of \( \delta X \) contained in \( X \) is an \( h' \)-point of \( X \). Let \( x \in X \) be a \( k' \)-point of \( \delta X \). Suppose \( A \subseteq X \) with \( x \in \text{cl}_X(A) \subseteq \text{cl}_{\delta X}(A) \). Then \( x \in \text{cl}_{\delta X}(A \cap B) \) for some compact subset \( B \) of \( \delta X \). Since \( A \cap B \subseteq X \cap B = H \), a hard set of \( X \) by 2.1 a), \( A \cap B = A \cap H \) and \( x \in X \cap \text{cl}_{\delta X}(A \cap H) = \text{cl}_X(A \cap H) \). Thus \( x \) is an \( h' \)-point of \( X \).

Finally, it is clear that for any property \( p \), every point of \( X - N_p(X) \) is a \( p' \)-point of \( X \). In particular, \( S_k(\delta X) \subseteq N_k(\delta X) = N_{rc}(X) \) by 2.3 c).

We may now state our principal results of this section.

3.6. Theorem. \( \delta X \) is a \( k' \)-space \( \iff \) \( X \) is an \( h' \)-space and there is some \( \delta X \)-open set \( V \) such that \( S_k(\delta X) \subseteq V \subseteq X \).

Proof. \( \Rightarrow \) If \( \delta X \) is a \( k' \)-space, then \( S_k(\delta X) = \phi \). It follows from 3.5 that \( S_h(X) = \phi \), whence \( X \) is an \( h' \)-space, and we may take \( V = \phi \).

\( \Leftarrow \) Let \( A \subseteq \delta X \). Then \( \text{cl}_{\delta X}((A) = \text{cl}_{\delta X}(A \cap X) \cup \text{cl}_{\delta X}(A - X) \). Let \( x \in S_k(\delta X) \) such that \( x \in \text{cl}_{\delta X}(A) \). Now \( x \in V \subseteq X \), so \( x \in \text{cl}_{\delta X}(\delta X - X) \).

Since \( \text{cl}_{\delta X}(A - X) \subseteq \text{cl}_{\delta X}(\delta X - X) \), \( x \notin \text{cl}_{\delta X}(A - X) \), whence \( x \in \text{cl}_{\delta X}(A \cap X) \cap X = \text{cl}_X(A \cap X) \). Since \( X \) is an \( h' \)-space, there is some \( H \in H(X) \) such that \( x \in \text{cl}_X(A \cap H) \subseteq \text{cl}_{\delta X}(A \cap \text{cl}_{\delta X}(H)) \). Since \( \text{cl}_{\delta X}(H) \) is compact, \( x \) is a \( k' \)-point of \( \delta X \), contradiction to \( x \in S_k(\delta X) \). Thus \( S_k(\delta X) \)
= \phi \text{ and } \delta X \text{ is a } k'\text{-space.}

3.7. Theorem. If \( X \) is an \( h \)-space and if \( k(\delta X) \) is Tichonov, then \( \delta X \) is a \( k \)-space.

Proof. We show first that the restriction \( \tau_X \) of the identity map \( \tau : k(\delta X) \to \delta X \) to \( X \) is a closed map and hence a homeomorphism. Suppose \( F \subseteq k(\delta X) \) is closed. Then \( F \cap \tau^{-1}[B] \) is closed for every compact \( B \subseteq \delta X \). Since the restriction of \( \tau \) to the compact subset \( \tau^{-1}[B] \) of \( k(\delta X) \) is a homeomorphism and \( \tau \) is one-to-one, \( \tau[F \cap \tau^{-1}[B] \cap \tau^{-1}[X]] = [\tau[F] \cap X] \cap [B \cap X] \) is closed in \( X \). So by 2.1 a), \( \tau[F] \cap X = \tau_X[F \cap X] \) meets every hard subset of \( X \) in a closed set. Since \( X \) is an \( h \)-space, \( \tau_X \) is a homeomorphism. For the remainder of the proof, we identify \( X \) and \( \tau^{-1}[X] \).

Suppose \( H \subseteq X \) is hard. Then \( cl_{\delta X}(H) \) is compact since \( \tau \) is a homeomorphism on compact sets. Now by definition, \( H \subseteq cl_{k(\delta X)}(H) \subseteq \tau^{-1}[cl_{\delta X}(H)] \). Since \( \tau \) is one-to-one, if the last two sets were unequal, \( \tau[cl_{k(\delta X)}(H)] \) would be a proper closed subset of \( cl_{\delta X}(H) \). Thus \( cl_{k(\delta X)}(H) = \tau^{-1}[cl_{\delta X}(H)] \). Finally, note that \( k(\delta X) = \tau^{-1}[\delta X] = \tau^{-1}[\bigcup \{ cl_{\delta X}(H) : H \in H(X) \}] = \bigcup \{ \tau^{-1}[cl_{\delta X}(H)] : H \in H(X) \} \subseteq cl_{k(\delta X)}(X) \subseteq k(\delta X) \). Hence \( X \) is dense in \( k(\delta X) \).

Since \( k(\delta X) \) is a Tichonov space, the identity map \( \tau_X \) of \( X \) into \( \delta X \) has a continuous extension \( \beta \tau_X : \beta(k(\delta X)) \to \beta \delta X \). So by [GJ, 6.12 – 6.13], \( \beta \tau_X \) is a homeomorphism. Clearly it sends \( k(\delta X) \) onto \( \delta X \). So \( \delta X \) is a \( k \)-space.

We do not know whether the assumption \( k(\delta X) \) is Tichonov in 3.7 is needed. The converse to 3.7 can fail, as shown by Example 5.5. However, we do have a partial converse.

3.8. Lemma. If \( N_{rc}(X) \) is a \( k \)-space and if every point of \( cl_X[X - N_{rc}(X)] \) is an \( h' \)-point of \( X \), then \( X \) is an \( h \)-space.

Proof. Let \( A \subseteq X \) be such that \( A \cap H \) is closed for every hard set \( H \) of \( X \).
Note that \( \text{cl}(A) = \text{cl}[A \cap N_{rc}(X)] \cup \text{cl}[A \cap N_{rc}(X)] \).

Observe first that \( \text{cl}_X[A \cap N_{rc}(X)] = A \cap N_{rc}(X) \). Every compact set \( B \) of \( N_{rc}(X) \) is hard in \( X \), so \( B \cap A \) is closed. Thus \( A \cap N_{rc}(X) \) is closed (in \( N_{rc}(X) \), therefore in \( X \)).

Since \( \text{cl}_X[A - N_{rc}(X)] \subseteq \text{cl}_X[X - N_{rc}(X)] \), \( \text{cl}_X[A - N_{rc}(X)] \) consists of \( h' \) points of \( X \), and thus it is contained in \( A \). The result now follows from 3.4.

3.9. Theorem. If every point of \( \text{cl}_X[X - N_{rc}(X)] \) is an \( h' \)-point of \( X \), then \( \delta X \) is a \( k \)-space \( \iff \) \( X \) is an \( h \)-space and \( k(\delta X) \) is Tikhonov.

Proof. Since being a \( k \)-space is closed hereditary, we have that \( N_k(\delta X) = N_{rc}(X) \) is a \( k \)-space. The result now follows from 3.7 and 3.8.

3.10. Theorem. If \( X \) is an \( h' \)-space, then \( \delta X \) is a \( k' \)-space.

Proof. Suppose \( f \) is any real valued function on \( \delta X \) such that the restriction \( f_B \) of \( f \) to each compact subset \( B \) of \( \delta X \) is continuous. Then the restriction \( f_X \) of \( f \) to \( X \) is continuous on \( B \cap X \) for each compact subset \( B \) of \( \delta X \), hence to each hard subset of \( X \). By assumption, \( f_X \in C(X) \). Let \( \delta f_X \) denote the restriction to \( \delta X \) of the Stone extension \( f_B \) of \( f_X \) into the two point compactification of \( X \).

Clearly \( f \) and \( \delta f_X \) coincide on \( \text{cl}_{\delta X}(H) \subseteq \delta X \) for each hard \( H \subseteq X \). Hence \( \delta f_X = f \) is continuous.

The converse to 3.10 can fail, as shown by Example 5.5, below.

4. Some mapping theorems.

Let us refer to a continuous, onto function as a map. In [A], a map \( f : X \to Z \) is defined to be pseudo-open if for each \( y \in Z \) and each open \( U \) of \( X \) such that \( f^{-1}(y) \subseteq U \), we have \( y \in \text{int}_Z f[U] \). It is shown there that every open map and every closed map is a pseudo-open map, and every pseudo-open map is a quotient map. It is noted in [E, 2.4P] that \( f \) is pseudo-open if and only if for each
A \subseteq Z \), the restriction of \( f \) to \( f^{-1}(A) \) is a quotient map. In connection with this and the result \([E, 2.4.6]\) that if \( f : X \to Z \) is a map and \( W \subseteq X \) is such that \( f^{-1}(W) = Z \), then \( f \upharpoonright W \) is quotient implies \( f \) is quotient, we remark:

4.1. Theorem. Let \( f : X \to Z \) be a map and \( W \subseteq X \) such that \( f^{-1}(W) = Z \). If \( f \upharpoonright W \) is pseudo-open, then \( f \) is pseudo-open.

Proof. Let \( y \in Z \) and consider \( f^{-1}(y) \subseteq U \), open in \( X \). Then \( f^{-1}(y) \cap W \neq \emptyset \) and \( f^{-1}(y) \cap W \subseteq U \cap W \). By assumption, \( y \in \text{int}_Z f^{-1}(U \cap W) \subseteq \text{int}_Z f(U) \). So \( f \) is pseudo-open.

We also note that if \( X \) is the free union of \([0, 1]\) and \([1, 2]\), \( Z = [0, 2] \) and \( f : X \to Z \) is the identity on each component, then \( f \) is pseudo-open, but \( f \upharpoonright [0, 1] \) is not. However, we can say:

4.2. Theorem. The restriction of a pseudo-open map to a saturated subset is pseudo-open.

Proof. A saturated subset of \( X \) is a subset of the form \( A = f^{-1}(f(A)) \). The result follows immediately from the characterization of a pseudo-open map as hereditarily quotient.

Arhangel'skiǐ shows \([A]\) that \( X \) is a \( k' \)-space if and only if \( X \) is the image of a locally compact space under a pseudo-open map. We can emulate a portion of his result for any \( p' \)-space.

4.3. Theorem. Every \( p' \)-space is the image of a locally-\( p' \) space under a pseudo-open map.

Proof. Let \( X \) be a \( p' \)-space and define \( Z \) to be the free union of the sets of \( F \subseteq \mathcal{F}_p \). Let \( \tau : Z \to X \) be such that for each \( F \in \mathcal{F}_p \), \( \tau \upharpoonright F = \text{id}_F \). Then \( Z \) is clearly locally-\( p' \). It remains to show that \( \tau \) is pseudo-open. Suppose not. Let \( x \in X \) and \( U \) be open in \( Z \) so that \( \tau^{-1}(x) \subseteq U \), but \( x \notin \text{int}_X \tau(U) \). Then \( x \in \overline{X} \left[ X - \tau(U) \right] \). Since \( X \) is \( p' \), there is some \( F \in \mathcal{F}_p \) with \( x \in \overline{X} \left( F \cap \left[ X - \tau(U) \right] \right) \). Write \( F \) as \( F' \) considered as a component of
4.4. Corollary. Every $p'$-space and every $re'$-space is the image of a locally realcompact space under a pseudo-open map.

The converse of Arhangel'skii's characterization can also be generalized, but only at the cost of adding a qualification to the map which may be difficult to achieve in practice.

4.5. Theorem. Let $X$ be a $p'$-space and $f : X \rightarrow Z$ be a map. If $f$ is pseudo-open and if for each $F \in \mathcal{F}_p(X)$, there is some $B \in \mathcal{F}_p(Z)$ such that $f[F] \subseteq B$, then $Z$ is a $p'$-space.

Proof. Let $y \in Z$ and $A \subseteq Z$ such that $y \in \text{cl}_Z[A]$. Then $f^+(y) \cap \text{cl}_Xf^+[A] \neq \emptyset$. Otherwise $f^+(y) \subseteq U = X - \text{cl}_Xf^+[A]$, so $y \in \text{int}_Zf[U]$ which is disjoint from $A$, a contradiction. Let $x \in f^+(y) \cap \text{cl}_Xf^+[A] \subseteq \text{cl}_Xf^+[A]$. Since $X$ is a $p'$-space, there is an $F \in \mathcal{F}_p(X)$ for which $x \in \text{cl}_X[F \cap f^+[A]]$. Let $B \in \mathcal{F}_p(Z)$ such that $f[A] \subseteq B$. Then $y = f(x) \in f[\text{cl}_X(F \cap f^+[A])] \subseteq \text{cl}_Z[f(F \cap f^+(A))] \subseteq \text{cl}_Z[A \cap f(F)] \subseteq \text{cl}_Z(A \cap B)$. Hence $y$ is a $p'$ point of $Z$ and $Z$ is a $p'$-space.

The situation is analogous for $p$-spaces. Notice

4.6. Lemma. If $\mathcal{F}$ is any family of regular closed sets whose interiors cover $X$, then $\mathcal{F}$ determines the topology of $X$ (in that a subset of $X$ is closed if and only if it intersects each $F \in \mathcal{F}$ in a closed subset).

4.7. Theorem. $X$ is a $p$-space $\iff$ there is a locally-$p$ space $Z$, a quotient map $f : Z \rightarrow X$ and a regular closed family $\mathcal{F}$ of sets whose interiors cover $Z$ such that for each $F \in \mathcal{F}$, $f[F] \in \mathcal{F}_p(X)$.

Proof. $\Rightarrow$ Let $Z$ be the free union of the sets of $\mathcal{F}_p$ and $f : Z \rightarrow X$ be
given by \( f \upharpoonright F = \text{id}_F \) for each \( F \in \mathcal{T}_p \). It is seen in the usual way that \( Z \) is locally-\( p \) and \( f \) is quotient. Since each \( F \in \mathcal{T}_p \) is both open and closed as a component of \( Z \), the remaining condition is also verified.

Let \( A \subseteq X \) be such that \( A \cap F \) is closed for every \( F \in \mathcal{T}_p \). For each \( B \in \mathcal{F} \), \( A \cap f[B] \) is closed, thus \( f^\sim(A) \cap f^\sim(f[B]) \) is closed in \( Z \). Hence \( B \cap f^\sim(A) \cap f^\sim(f[B]) = B \cap f^\sim(A) \) is closed in \( Z \). Since by 4.6, \( \mathcal{F} \) determines the topology of \( Z \), \( f^\sim(A) \) is closed in \( Z \). Since \( f \) is quotient, \( A \) is closed in \( X \).

5. Examples.

Before presenting the examples promised above, we review some known topological constructions and develop some of their properties. If \( \gamma \) is any ordinal, let \( W(\gamma) \) denote the space of ordinals \( < \gamma \). Suppose \( X \) is any (Tikhonov) space and \( \omega_\alpha \) is the initial ordinal corresponding to the smallest regular cardinal such that \( |\omega_\alpha| > |\beta X| \). Let \( \eta(X) = W(\omega_\alpha + 1) \times \beta X - \{\omega_\alpha\} \times (\beta X - X) \). In \( [N] \), N. Noble shows that \( \eta(X) \) is always a pseudocompact \( k \)-space. Since \( \{\omega_\alpha\} \times X \) is closed in \( \eta(X) \), it follows that any (Tikhonov) space can be a closed subspace of a \( k \)-space.

5.1. Theorem. \( \eta(X) \) is a \( k \)-space if and only if \( X \) is a \( k \)-space.

Proof. Since \( X \) is homeomorphic to the closed subspace \( X' = \{(\omega_\alpha, x) : x \in X \} \) of \( \eta(X) \), the necessity follows from the fact that being a \( k \)-space is closed hereditarily.

Suppose \( X \) is a \( k \)-space and \( A \subseteq \eta(X) \) meets every compact subset of \( \eta(X) \) in a closed set. By Lemma 3.3, \( cl(A) - A \subseteq S_k(\eta(X)) \subseteq X' \). If \( t \in cl(A) - A \), then \( t \in cl_{\eta(X)}(A \cap X') \) or \( t \in cl_{\eta(X)}(A - X') \). If the former holds, then \( t \in A \) since \( X' \) is a \( k \)-space. So we may assume that \( t \in cl_{\eta(X)}(A - X') \) and \( t = (\omega_\alpha, x) \) for some \( x \in X \).

Let \( V \) be an \( X \)-open neighborhood of \( x \). Then there is some \( y \in V \) for which \( \{|(A - X') \cap \{(\gamma, y) : \gamma < \omega_\alpha \}| = |\omega_\alpha| \). Suppose not. Then for
every \( y \in V \), \( (A - X') \cap \{ (\gamma, y) : \gamma < \omega_\alpha \} \) \( \subseteq \{ \omega_\alpha \} \). For each \( y \), let \( \delta_y = \sup \{ \gamma : (\gamma, y) \in (A - X') \} \), and \( \delta_0 = \sup \{ \delta_y : y \in V \} \). Since \( |\beta X| < |\omega_\alpha| \) and \( \omega_\alpha \) is regular, \( \delta_0 < \omega_\alpha \). But then \( [\delta_0 + 1, \omega_\alpha] \times V \) is an open neighborhood of \( t \) which misses \( A - X' \), contradiction.

Thus, for each \( X \)-open neighborhood \( V \) of \( x \), we have some \( y_v \in V \) which has a cofinal subset of \( W(\omega_\alpha) \times \{ y_v \} \) lying in \( A - X' \). It follows that \( (\omega_\alpha, y_v) \in cl_{\eta(X)}[(A - X') \cap (W(\omega_\alpha + 1) \times \{ y_v \})] \). Since \( W(\omega_\alpha + 1) \times \{ y_v \} \) is compact, \( A \cap [W(\omega_\alpha + 1) \times \{ y_v \}] \) is closed and \( (\omega_\alpha, y_v) \in A \).

Since this is true for each \( X \)-open \( V(x) \), it follows that \( (\omega_\alpha, x) \in cl_{\eta(X)}[A \cap X'] \). But we have already noted that \( A \cap X' \) is closed. Thus \( t \in A \), and \( \eta(X) \) is a \( k \)-space.

5.2. Example. If \( X \) is not a \( k \)-space, then since \( \eta(X) \) is pseudocompact, it is an \( h \)-space that is not an \( h \)-space.

Suppose \( X \) is a \( k \)-space that is not a \( k' \)-space (see [A, Example 3.1] or Example 5.4 below). By Theorem 5.1, \( \eta(X) \) is a pseudocompact \( k \)-space that is not a \( k' \)-space, since \( \eta \) by [A, Proposition 3.1] being a \( k' \)-space is a closed hereditary property. Hence \( \eta(X) \) is an \( h \)-space that is not an \( h' \)-space.

We shall make use of the following theorem of W. Comfort, [C, 2.4]:

5.3. Theorem. If \( Z \) is a \( k \)-space, \( vX \) is locally compact, and \( X \times Z \) has non-measurable cardinal, then \( \nu(X \times Z) = \nu X \times \nu Z \).

Our next example will show that the converse of Theorem 2.8 fails to hold. In what follows, we abbreviate \( W(\omega_1) \) by \( W \) and \( W(\omega_1 + 1) \) by \( W^* \).

5.4. Example. Let \( Z = W \times N \) denote the free union of countably many copies of \( W \). By 5.3, \( \nu Z = W^* \times N \), so \( \nu Z - Z = \{ \omega_1 \} \times N \) fails to be closed in \( \beta Z - Z \). But \( Z \) is locally compact, so \( N_{re}(Z) \) is empty and hence pseudocompact, so (a) of Theorem 2.8 holds.

For each positive integer \( n \), let \( S_n \) be a compact subset of \( W \times \{ x \} \) and let \( S = \cup_1^{\infty}(S_n \times \{ n \}) \). Assume \( S \neq \phi \). Then \( S \) is a \( \sigma \)-compact, hence realcompact...
set which is $C$-embedded in $Z$.

Conversely, if $S$ is a realcompact, $C$-embedded subset of $Z$, then by [GJ, 8.10 (a)], $S$ is closed in $vZ$, and hence in $Z$. Thus for each $n$, $S_n = S \cap \{ w \times \{ n \} \}$ is closed in the compact space $W^* \times \{ n \}$, and $S = \bigcup_1^\infty (S_n \times \{ n \})$ is $\sigma$-compact. Moreover, since each $S_n$ is bounded away from $(\omega_1, n)$ in $W^* \times \{ n \}$, $S$ is bounded away from $K_Z$ in $Z \cup K_Z$. Therefore $S$ is closed in $Z \cup K_Z$, i.e. $S$ is a hard set of $Z$. Thus, both (a) and (b) of Theorem 2.8 hold and its hypothesis fails to hold.

Our final example shows that neither the converse of Theorem 3.7 nor that of Theorem 3.10 hold.

5.5 Example. Let $Z$ be the space of Example 5.4, let $S = K_Z - vZ = cl_{\beta Z}(\{ \omega_1 \} \times N) - (\{ \omega_1 \} \times N)$, and let $X = Z \cup S$. As in Example 5.4, it is easy to see that $K_X = K_Z = cl_{\beta X}(\{ \omega_1 \} \times N)$. So $\delta X = \beta X - (K_X - X) = \beta X - (\{ \omega_1 \} \times N)$ is a $G_\delta$ set in $\beta X$. By [A, Theorem 3.8, Corollary 1], $\delta X$ is a $k$-space. We shall show that $X$ is not an $h_\beta$-space.

Let $f$ be a real valued function on $X$ be defined by letting $f(x) = m$ if $x \in W \times \{ m \}$ for some $m \in N$, and $f(x) = 0$ if $x \in S - Z$. Since $X \cup K_X = (W^* \times N) \cup S$, each hard subset of $X$ is the union of a $\sigma$-compact subset of $Z$ and a compact subset of $S$. It follows that the restriction of $f$ to each hard subset of $X$ is continuous, while $f$ fails to be continuous at any point of $S$. So $X$ is not an $h_\beta$-space, and hence not an $h$-space.

It follows from Theorem 3.6 that $\delta X$ is not a $k^*$-space. Indeed, one may verify directly that no point of $S$ is either a $k^*$-point of $\delta X$ or an $h^*$-point of $X$.

References


