f-rings, subdirect products of totally ordered rings, and the prime ideal theorem

by David Feldman¹ and Melvin Henriksen²

¹ Department of Mathematics, University of New Hampshire, Durham, N.H. 03824, U.S.A.
² Department of Mathematics, Harvey Mudd College Claremont, CA 91711, U.S.A.

Communicated by Prof. A.C. Zaanen at the meeting of September 28, 1987

ABSTRACT

An f-ring is a lattice-ordered ring \( R \) in which \( x^+ \land x^- x^+ = x^+ \land x^+ x^- = 0 \) for all \( x \in R \). It is shown that every f-ring is a subdirect product of totally ordered rings iff the Prime Ideal Theorem for Boolean algebras holds. Moreover, every f-ring is an \( l \)-homomorphic image of a subdirect product of totally ordered rings.

1. INTRODUCTION

In 1956, in their classical paper [BP] on lattice-ordered rings (or \( l \)-rings), G. Birkhoff and R.S. Pierce defined an f-ring to be an \( l \)-ring \( R \) in which:

\[
(1) \quad a \land b = 0 \quad \text{and} \quad c \geq 0 \quad \text{imply} \quad a \land bc = a \land cb = 0.
\]

It is routine and well known that (1) is equivalent to the identities:

\[
(2) \quad x^+ \land x^- y^+ = 0 = x^+ \land y^+ x^-
\]

where \( x^+ = x \lor 0 \) and \( x^- = (-x) \lor 0 \).

See [BKW, 9.1.4].

It is shown in [BP] with the aid of the axiom of choice (AC) that \( R \) is an f-ring if and only if:

\[
(3) \quad R \text{ is a subdirect product of totally ordered rings.}
\]

Indeed, many authors have taken (3) as the definition of f-ring since any identity that holds in all totally ordered rings holds in any \( l \)-ring satisfying (3).
For example, in [BKW] an \( f \)-ring is defined by (3), but in this note the definition (1) or its equivalent (2) will be used.

We will say that a theorem holds in \( ZF \) if it is derivable with the aid of the Zermelo-Fraenkel axioms for set theory, and that it holds in \( ZFC \) if it is derivable from these latter axioms together with \( AC \). While most of the papers written about \( f \)-rings assume and use \( AC \) freely, C. Huijsmans and B. de Pagter (among others) have written a number of papers about \( f \)-rings working exclusively in \( ZF \). An incomplete list of such papers is [H deP 1], [H deP 2], and [BH].

The purpose of this note is to show that (1) implies (3) if and only if the Prime Ideal Theorem for Boolean algebras holds, and that in \( ZF \) the class of \( f \)-rings coincides with the class of \( l \)-rings that are \( l \)-homomorphic images of subdirect products of totally ordered rings. In particular, we produce a model of \( ZF \) in which there is an \( f \)-ring with an \( l \)-homomorphic image that is not a subdirect product of totally ordered rings, and conversely, we show that all \( f \)-rings are \( l \)-homomorphic images of subdirect products of totally ordered rings.

2. \( f \)-RINGS VS. SUBDIRECT SUMS OF TOTALLY ORDERED RINGS

Recall that an \( l \)-ring \( R = \langle +, - \times, \wedge, \vee \rangle \) is a ring \( R = \langle +, - \rangle \) that is a lattice \( R(\vee, \wedge) \) such that \( a, b \geq 0 \) implies \( a + b \geq 0 \) and \( ab \geq 0 \). For background, see [BP] or [BKW]. An \( l \)-homomorphism, \( \phi : R \rightarrow S \) of an \( l \)-ring \( R \) into an \( l \)-ring \( S \) is a ring homomorphism such that for all \( a, b \in R \), \( \phi(a \vee b) = \phi(a) \vee \phi(b) \) and \( \phi(a \wedge b) = \phi(a) \wedge \phi(b) \).

The kernel \( \ker \phi \) of an \( l \)-homomorphism is called an \( l \)-ideal; it is a ring ideal of \( R \) such that if \( |a| \leq |b| \) and \( b \in \ker \phi \), then \( a \in \ker \phi \). Conversely, if \( I \) is an \( l \)-ideal of the \( l \)-ring \( R \) then \( R/I = \{ a + I : a \in R \} \) is an \( l \)-ring if we let \( (a + I) \vee (b + I) = a \vee b + I \) and \( (a + I) \wedge (b + I) = a \wedge b + I \). If \( \phi : R \rightarrow S \) is an \( l \)-homomorphism and \( S \) is totally ordered then clearly

\[
\phi(a \wedge b) = 0 \text{ implies } \phi(a) = 0 \text{ or } \phi(b) = 0.
\]

Recall that a filter \( \mathcal{F} \) on a nonempty set \( D \) is a family of subsets of \( D \) closed under finite intersection that contains every superset of each of its elements and does not contain the empty set. A filter \( \mathcal{F} \) is called fixed or free according as every element of \( \mathcal{F} \) has a point in common or not. An ultrafilter is a maximal filter. All supersets of a singleton of \( D \) form a fixed ultrafilter, and there are models of \( ZF \) in which a countably infinite set has no free ultrafilters [Je, Problem 5.24].

Let \( C(D) \) denote the \( l \)-ring of all real-valued functions on a nonempty set \( D \).

For any \( f \in C(D) \) it is clear that \( f^+ f^- = f^- f^+ = 0 \), so \( C(D) \) is an \( f \)-ring by (2). For \( f \in C(D) \), let \( Z(f) = \{ x \in D : f(x) = 0 \} \). It is well-known and easily seen that \( I \) is a proper ideal of \( C(D) \) if and only if \( Z(I) = \{ Z(f) : f \in I \} \) is a filter on \( D \).

Consider each of the following assertions.

The Prime Ideal Theorem (PIT). Every Boolean algebra contains a (proper) prime ideal.
The Ultrafilter Theorem (UT). Every filter on a set $D$ is contained in an ultrafilter on $D$.

The Compactness Theorem (CT). Suppose $\mathcal{L}$ is a language of first order logic, and let $\Sigma$ denote a set of sentences of $\mathcal{L}$. If every finite subset of $\Sigma$ has a model, then $\Sigma$ has a model.

In [Je, Theorem 2.2] it is shown that:

(5) In $ZF$, the statements $PIT$, $UT$, and $CT$ are equivalent.

Clearly $UT$ holds in $ZFC$ and in [Je, Section 7.1] it is shown that $PIT$ does not imply $AC$. So the equivalent assertions in (5) are properly weaker than $AC$.

An $l$-ideal $M$ of an $l$-ring $R$ is called semi-maximal if there is an $a \in R$ such that $M$ is maximal with respect to the property of not containing $a$. We include a proof of the following well-known lemma to assure the reader that it holds in $ZF$. If $S \subseteq R$, let $\langle S \rangle$ denote the smallest $l$-ideal containing $S$.

1. LEMMA. If $M$ is a semi-maximal $l$-ideal of an $f$-ring $R$, then $R/M$ is totally ordered.

PROOF. If $R/M$ fails to be totally ordered, there is an $x \in R$ such that neither $x^+$ nor $x^-$ is in $M$. Thus, if $M$ is maximal with respect to not containing $a$, then

$$a \in (M+\langle x^+ \rangle) \cap (M+\langle x^- \rangle) \subseteq (M+\langle x^+ \rangle) \cap \langle x^- \rangle).$$

By (2), since $x^+ \wedge x^- = 0$, we have $\langle x^+ \rangle \cap \langle x^- \rangle = 0$ in any $f$-ring, so $a \in M$. This completes the proof of the lemma.

The next lemma will also be used below.

2. LEMMA. If $R$ is a countable $l$-ring and $0 \neq a \in R$, there is an $l$-ideal $M$ of $R$ maximal with respect to not containing $a$.

PROOF. The conclusion is obvious if $R$ is finite, so we may assume that $R$ is countably infinite. Let $\{x_n\}$ be an enumeration of $R \setminus \{a\}$ such that $x_0 = 0$. (The set of enumerations of $R \setminus \{a\}$ is nonempty and $AC$ is not needed to choose one of them.) Let $y_0 = x_0$ and for $n \geq 0$, define $y_1, y_2, \ldots$, successively by letting $y_{n+1} = y_n$ if $a \in \langle y_0, y_1, \ldots, y_n, x_{n+1} \rangle$ and $y_{n+1} = x_{n+1}$ otherwise. Clearly if $M$ is the $l$-ideal generated by $\{y_n : n = 0, 1, 2, \ldots\}$, then $M$ is an $l$-ideal maximal with respect to not containing $a$.

The proof of the next lemma may be compared with the proof-outline that (iii) $\Rightarrow$ (iv) in [Je, p. 18]. In it we make use of the fact that the sub-$f$-ring $T$ generated by a finite set $F$ of elements of an $f$-ring $R$ may be shown to be countable in $ZF$. To see this, note that $T$ is a subset of the collection of "words" formed from $F$, its set of negatives, and the operation symbols of $R$. The words of each fixed length form a finite set which can be listed lexi-
graphically; and \( T \) can be enumerated by a Cantor diagonal process upon eliminating any elements that duplicate ones listed before or which make no sense.

No claim is made that this is a constructive procedure; it may or may not be possible to tell effectively if two elements of an \( f \)-ring as described above are equal or if \( T \) is finite or infinite. None-the-less it follows in \( ZF \) that \( T \) is countable.

3. LEMMA. \( \text{Suppose } CT \text{ holds. If } R \text{ is an } f \text{-ring and } a \neq 0 \text{ is in } R, \text{ then there is a } l \text{-homomorphism } \phi \text{ of } R \text{ into a totally ordered ring such that } \phi(a) \neq 0. \)

PROOF. Let \( \{ y : u \in R \} \) denote a family of distinct constants and let \( I \) denote a unary predicate. \( \Sigma \) will denote the following family of sentences, none of which contain quantifiers.

(a) All equations in the constants \( y \) which correspond to equations that hold in \( R \).

(b) \( I(g) \) and \( \neg I(g) \)

(c) For each \( u, u_1, u_2 \) in \( R \)

(a) \( I(u_2) \) \( \Rightarrow I(u_1 - u_2) \)

(b) \( I(u_1) = I(u \ y_1) \) \( \text{and} \ I(u_1 \ u) \)

(c) \( I(u_1) \) and \( |y| \vee |y_1| = |y_1| \Rightarrow I(u) \) (Recall that \( |u| = u^+ + u^- \)

(d) \( I(u^+) \) or \( I(u^-) \)

Let \( \Sigma' \) denote a finite subset of \( \Sigma \) and let \( T \) denote the sub-\( f \)-ring of \( R \) generated by the finite set of elements of \( R \) that correspond to the constants in \( \Sigma' \). Since \( T \) is countable, Lemmas 1 and 2 provide an \( l \)-homomorphism of \( T \) onto a totally ordered ring whose kernel is a prime \( l \)-ideal \( P \) not containing \( a \). Thus \( \Sigma' \) has a model \( \mathcal{M} \). By \( CT \), there is a model \( \mathcal{M} \) of \( \Sigma \). By (a), the subset of \( \mathcal{M} \) that interprets \( \{ y : u \in R \} \) is an \( f \)-ring which may be identified with \( R \). The predicate \( I \) restricted to this subset yields a prime \( l \)-ideal of \( R \) which does not contain \( a \). This prime \( l \)-ideal is the kernel of the desired \( l \)-homomorphism.

REMARK. \( CT \) does not imply the theorem of Krull that every commutative ring \( R \) with identity element contains a maximal ideal. For, as was shown by W. Hodges in [H], Krull’s theorem implies \( AC \), which is stronger than \( CT \) as noted above. The argument given in the proof of Lemma 3 does not yield Krull’s theorem, even though the notion of a maximal ideal of \( R \) may be captured by a unary predicate which “says” that \( I \) is an ideal of \( R \) and

\[ I(g) \text{ or } \exists b I(1 - ba) \text{ for each } a \in R. \]

The argument does produce a model \( \mathcal{M} \) that contains \( R \) and satisfies (6) for each \( a \in R \), but in general, \( b \) may have to come from \( \mathcal{M} \setminus R \). So \( R \) may not be a submodel of \( \mathcal{M} \). The culprit here is the existential quantifier in (6). This is similar to the phenomena that the intersection of a subring and a maximal ideal of a commutative ring may not be a maximal ideal of the subring. Indeed \( \mathcal{M} \) may be larger than \( R \) and there is no guarantee that it is a ring at all. This point
is elided in Jech’s discussion of why \textit{PIT} follows from \textit{CT} in [Je, p. 18]. (Also his set \( \Sigma \) of sentences seems to be too small).

The first of our two theorems is given next.

4. \textbf{Theorem.} The \textit{Prime Ideal Theorem} holds if and only if every \( f \)-ring is a subdirect product of totally ordered rings.

\textbf{Proof.} Assume first that \textit{PIT} holds in which case \textit{CT} holds as noted in (5). By Lemma 3, for each \( a \neq 0 \) in \( R \), there is an \( l \)-homomorphism \( \phi_a \) of \( R \) into a totally ordered ring such that \( a \notin \ker \phi_a \). Thus if \( \Phi \) is the collection of all \( l \)-homomorphisms that send \( R \) into a totally ordered ring, then

\[ \cap \{ \ker \phi : \phi \in \Phi \} = \{0\}, \]

so \( R \) is a subdirect product of totally ordered rings.

Suppose conversely that every \( f \)-ring is a subdirect product of totally ordered rings. We will show that \( UT \) holds and hence \textit{PIT} holds by (5). Let \( D \) be an infinite set and let \( \mathcal{F} \) be a filter on \( D \). As noted above, \( C(D) \) is an \( f \)-ring and \( I = \{ f \in C(D) : Z(f) \in \mathcal{F} \} \) is an \( l \)-ideal of \( C(D) \). Thus \( R = C(D)/I \) is an \( f \)-ring by (2). Let \( \phi : C(D) \to C(D)/I \) be the natural \( l \)-homomorphism given by \( \phi(f) = f + I \). Since \( R \) is an \( f \)-ring, there is an \( l \)-homomorphism \( \psi : R \to T \) onto a nonzero totally ordered ring \( T \). Then \( \psi \circ \phi \) is an \( l \)-homomorphism of \( C(D) \) onto \( T \) such that \( I = \ker \phi \subset \ker (\psi \circ \phi) \), and clearly \( Z(\ker (\psi \circ \phi)) \) is a filter on \( D \) containing \( \mathcal{F} \). Since \( T \) is totally ordered, \( \ker (\psi \circ \phi) \) satisfies (4). So if \( Z(f) \cup Z(g) = z([f] \cup [g]) \) is in \( \mathcal{U} \), then \( Z(f) \) or \( Z(g) \) is in \( \mathcal{U} \). As noted in [GJ, 2.14], this implies that \( \mathcal{U} \) is an ultrafilter. So \( UT \) and hence \textit{PIT} holds.

As noted above a number of results proved earlier with the aid of (3) are proved in [BH], [H deP 1], and [H deP 2] using only the definition of \( f \)-ring given in (1). As the last theorem shows, this amounts to avoiding the use of \textit{PIT}. Our final theorem will show that even if an \( f \)-ring fails to be a subdirect product of totally ordered rings, it still must be an \( l \)-homomorphic image of such a subdirect product.

Since they are defined within the class of \( l \)-rings by means of identities (2), the class of \( f \)-rings is a variety. In particular, every sub-\( l \)-ring, and every \( l \)-homomorphic image of an \( f \)-ring is an \( f \)-ring, as is any subdirect product of \( f \)-rings. Whether or not the class of subdirect products of totally ordered rings is a variety is independent of \( ZF \) as Theorem 4 shows. Let \( V(\mathcal{F}) \) denote the class of \( l \)-rings that satisfy all identities that hold in every totally ordered ring. By a well-known theorem of G. Birkhoff (whose proof does not depend on \textit{AC}) \( V(\mathcal{F}) \) consists of all \( l \)-homomorphic images of subdirect products of totally ordered rings. Moreover, in \( ZF \) every \( l \)-ring in a variety is the \( l \)-homomorphic image of a free \( l \)-ring in the variety; See [Ja, Sections 2.7–2.10]. The following also holds in \( ZF \).

5. \textbf{Theorem.} The variety of \( f \)-rings and the variety \( V(\mathcal{F}) \) of \( l \)-homomorphic images of subdirect products of totally ordered rings coincide.
PROOF. We will show that both varieties of \( l \)-rings satisfy the same identities. Clearly each element of \( V(\mathcal{D}) \) is an \( f \)-ring. Assume now that \( w = 0 \) is an identity which holds in each element of \( V(\mathcal{D}) \) and which contains \( n \) variables, for some positive integer \( n \). It suffices to show that \( w = 0 \) holds in the free \( f \)-ring \( F \) on \( n \) generators and hence in all \( f \)-rings. If \( w \) is nonzero in \( F \), then by Lemmas 1 and 2, there is an \( l \)-homomorphism \( \phi \) of \( F \) onto a totally ordered ring \( T \) such that \( \phi(w) \neq 0 \). But this contradicts the fact that \( w \) is an identity that holds in \( V(\mathcal{D}) \) and hence in any totally ordered ring.

Since an abelian \( l \)-group can be regarded as an \( f \)-ring with trivial multiplication, our results apply to abelian \( l \)-groups as well as to \( f \)-rings.

For more information about the relationship between the existence of certain kinds of ideals in algebraic systems, \( AC \), and \( PIT \), see [B].

REFERENCES


