1. Introduction

When George Martinez invited me, close to a year ago to give a talk on rings of continuous functions at a conference on ordered algebraic systems in an exotic location, I felt honored and elated. The accompanying feeling of euphoria stayed with me until I started to think seriously about what I could say to an audience containing people thoroughly familiar with the Gillman-Jerison text [GJ1] and many of the ensuing developments since 1960 – a lot of them due to mathematicians to whom I would be talking. I began to wonder if I would be doing the equivalent of carrying coals to Curaçao in August.

With no further introspection, this article will be an incomplete survey of developments in the study of algebraic aspects of rings of continuous functions since the publication of [GJ1] in 1960. As usual, let $C(X)$ denote the ring (with the usual coordinatewise operations of addition and multiplication) of continuous real-valued functions on a topological space $X$. Under suitable restrictions on $X$, the ring $C(X)$ determines $X$ (This is the case, in particular if $X$ is a compact (Hausdorff) space).
Studying the ring \( C(X) \) makes one ask questions about the underlying space that would not arise if one looks at topological spaces in more classical ways. Most of the research activity since 1960 has been concerned with the underlying space \( X \) rather than the algebraic structure of the ring \( C(X) \). Below I will emphasize the latter rather than the former. As L. Gillman reminded us in the title of [G], “Rings of Continuous Functions are Rings.” I will try to revive interest in some old unsolved problems, and will state some new ones. I will give brief synopses of many papers and leave it to the interested reader to examine the details. Some historical remarks are made with no pretense of giving a complete history.

I begin with a summary of those parts of [GJ1] essential to understanding what follows. Despite its venerable age, this excellently written book must be read by any serious student of \( C(X) \). Its results have been generalized and improved upon, and many of these improvements are recorded books such as [CN], [PW], [Wa], and [We], but [GJ1] is far from obsolete.

Section 1 consists mainly of a review of those parts of [GJ1] needed to set the stage for the sequel and to describe the nature of the subject, but some material that appeared after 1960 is included. In particular, some mention is made of R. L. Blair’s work on \( z \)-embedding, and the work of A. Stone and W. Comfort on normality and local compactness not being preserved in going from \( X \) to \( vX \) is mentioned.

Section 2 is devoted in part to the problems involved in translating topological concepts into algebraic language and vice versa. The complicated nature of real-compact spaces seems to make this difficult. Attempts to obtain internal algebraic characterizations of a \( C(X) \) are reviewed with particular emphasis on such an attempt by A. Hager. All such characterizations seem to rely on the representation theorem
RINGS OF CONTINUOUS FUNCTIONS FROM AN ALGEBRAIC POINT OF VIEW

for Φ-algebras as algebras of extended real-valued functions on a compact space due to D. G. Johnson and me.

Section 3 consists mainly of a review of the efforts to classify H-fields (i.e., fields of the form \( C(X)/M \), where \( M \) is a maximal ideal of \( C(X) \), that contains \( \mathbb{R} \) properly. Through the use of model theory, set theory, and the theory of ultrafilters much more has been learned about H-fields than appeared in the Gillman-Jerison text, but much more remains to be done. Recent progress on integral domains that are homomorphic images of a \( C(X) \) is also described.

In a final section, I mention briefly a few other related topics: discontinuous homomorphisms of \( C(X) \) into a Banach algebra, other algebraic structures on \( C(X) \), spaces of minimal prime ideals of \( C(X) \), and covers and completions.

I have tried to give credit where it is due, but I refer to books and monographs in preference to original sources.

1. Pre-requisites; setting the stage.

By and large, the notation and terminology of [GJ1] will be used unless it conflicts with the by now standard notation used by workers in lattice-ordered rings, in which case the terminology of [BKW] is adopted. The reader's attention will be called to conflicts in terminology as they arise. A Tychonoff space is a subspace of a compact (Hausdorff) space. (In [GJ 1], such spaces are called completely regular). A space \( X \) is called quasicompact if every open cover of \( X \) has a finite subcover, and a Hausdorff quasicompact space is compact. \( X \) is a Tychonoff space if and only if whenever \( F \subset X \) is closed and \( x \notin F \), there is \( f \in C(X) \) such that \( f(x) = \{0\} \) and \( f[F] = \{1\} \). It was known to both M.H. Stone and E. Čech that for each topological space \( X \), there is
a Tychonoff space $X'$ such that $C(X)$ and $C(X')$ are isomorphic. (See the historical
notes for Chapter 3 of [GJ1]). Since we regard topological spaces as the domain
of continuous real-valued functions, it will be assumed henceforth that $X$ denotes a
Tychonoff space unless the contrary is stated explicitly.

For $f \in C(X)$ the zero set of $F$ is $Z(f) = \{ x \in X : f(x) = 0 \}$, and $\text{coz } f = X \setminus Z(f)$
is called the cozero set of $f$. The set $Z(X) = \{ Z(f) : f \in C(X) \}$ is closed under
finite union (since if $f_i \in C(X)$ for $1 \leq i \leq n$, then $\bigcup_{i=1}^n Z(f_i) = Z(f_1, f_2 \cdots f_n)$)
and under countable intersection (since if $f_i \in C(X)$ for $i \geq 1$, $f = \sum_{i=1}^{\infty} \frac{|f_i| \wedge 1}{2^i}$,
then $\bigcap_{i=1}^{\infty} Z(f_i) = Z(f)$). $C(X)$ is a commutative ring whose identity element is the
constant function $1$. Let $C^*(X)$ denote the set of bounded elements of $C(X)$
and note that $C^*(X)$ is a subring of $C(X)$. Observe that $f \in C(X)$ is invertible if and
only if $Z(f) = \emptyset$, but this need not be the case for the subring $C^*$. For if $N$ denotes
the (discrete) space of positive integers, and $j(n) = 1/n$ for $n \in N$, then $Z(j) = \emptyset$,
but $j$ has no inverse in $C^*(N)$.

A subspace $S$ of $X$ is said to be $C$-embedded (resp. $C^*$-embedded) in $X$ if the
map $f \to f|_S$ is an epimorphism of $C(X)$ onto $C(S)$ (resp. $C^*(X)$ onto $C^*(S)$). The
following theorem, established independently by M.H. Stone and E. Čech, is essential
to all that follows. See [GJ1, Chap. 6].

1.1 Theorem Every Tychonoff space $X$ is dense and $C^*$-embedded in a compact space
$\beta X$. Moreover $\beta X$ is unique in the sense that if $X$ is a dense, $C^*$-embedded subspace
of a compact space $K$, then there is a homeomorphism of $\beta X$ onto $K$ whose restriction
to $X$ is the identity map.

There are three ways to construct $\beta X$ each of which is more useful than the other
two for some purposes. E. Čech proceeded as follows. Let $I(X) = C(X) \cap [0,1]^X$
and let \( I_f \) denote a copy of \([0, 1]\) for each \( f \in I(X) \). By the Tychonoff theorem [GJ1, Chapter 6], \( P = \pi\{I_f : f \in I(X)\} \) is compact, and the map \( e : X \to P \) given by \( e(x) = f(x) \) is injective and bicontinuous. It follows that \( \beta X = Cl_P(e[X]) \) contains \( e[X] \) as a dense \( C^* \)-embedded subspace and is compact.

M.H. Stone's construction is an algebraic one. Let \( \mathcal{M} \) denote the family of maximal ideals of \( C^*(X) \). If \( M \in \mathcal{M} \), then \( C^*(X)/M \) is a Dedekind complete ordered field and hence is isomorphic to the real field \( \mathbb{R} \). For \( f \in C^*(X) \), let \( h(f) = \{ M \in \mathcal{M} : f \in M \} \) and let \( h^c(f) = M \setminus h(f) \). Taking \( \{h^c(f) : f \in C^*(X)\} \) as a base for a topology on \( \mathcal{M} \) (thereby getting what is called the Stone topology or hull-kernel topology on \( \mathcal{M} \)) makes it into a compact space. The map that sends \( x \in X \) to \( M_x = \{ f \in C^*(X) : f(x) = 0 \} \) is a homeomorphism of \( X \) onto a dense subspace of \( \mathcal{M} \), and, for each \( f \in C^*(X) \), the map \( \tilde{f} : \mathcal{M} \to \mathbb{R} \) such that \( \tilde{f}(M) = f + M \) is a continuous real-valued extension of \( f \) over \( \mathcal{M} \). Thus \( \beta X = \mathcal{M} \) has the properties announced in Theorem 1.

A collection \( \mathcal{F} \) of zerosets of \( X \) such that (i) \( \emptyset \notin \mathcal{F} \), (ii) \( Z_1, Z_2 \in \mathcal{F} \) implies \( Z_1 \cap Z_2 \in \mathcal{F} \), and (iii) \( Z_1 \in \mathcal{F} \), \( Z \in Z(X) \) and \( Z_1 \subseteq Z \) implies \( Z \in \mathcal{F} \), is called a \( \mathcal{Z} \)-filter, and a maximal \( \mathcal{Z} \)-filter is called a \( \mathcal{Z} \)-ultrafilter, (In case \( X \) is a discrete space, the prefix "\( \mathcal{Z} \) - " is usually dropped). If \( \mathcal{F} \) is a \( \mathcal{Z} \)-filter, then \( I(\mathcal{F}) = \{ f \in C(X) : Z(f) \in \mathcal{F} \} \) is an ideal of \( C(X) \) which is maximal if and only if \( \mathcal{F} \) is a \( \mathcal{Z} \)-ultrafilter.

The Stone topology on \( \mathcal{M} \) induces a topology on the family of \( \mathcal{Z} \)-ultrafilters on \( X \). If \( M \in \mathcal{M} \), and \( Z(M) = \{ Z(f) : f \in M \} \) is the corresponding \( \mathcal{Z} \)-ultrafilter on \( X \), we take a base for a topology on the space \( \mathcal{U}(X) \) of \( \mathcal{Z} \)-ultrafilters on \( X \) by taking \( \{ Z(M) : f \notin M \} : f \in C^*(X) \) as a base for a (compact) topology. A \( \mathcal{Z} \)-filter \( \mathcal{F} \) is called fixed or free according as \( \cap \mathcal{F} \) is nonempty or empty, and the space \( X \) is
homeomorphic to the dense subspace of \( U(X) \) if \( x \in X \) is mapped onto the (fixed) \( z \)-filter of zero sets containing \( x \). Similarly, an ideal \( I \) of \( C(X) \) is called \textit{fixed} or \textit{free} according as \( \cap Z(I) \) is nonempty or empty. The image of \( X \) under this mapping is dense and \( C^* \)-embedded in \( X \), so \( \beta X = U(X) \). This last construction is due to E. Hewitt [Hew] and L. Gillman and M. Jerison [GJ1, Chap 6]. The latter should be examined for more details and for a proof of the uniqueness of \( \beta X \).

If \( X \) is a compact space, then every maximal ideal of \( C(X) \) is fixed. That is, if \( M \) is a maximal ideal of \( C(X) \), there is a unique \( x \in X \) such that \( M = M_x = \{ f \in C(X) : f(x) = 0 \} \) and the map \( x \to M_x \) is a homeomorphism of \( X \) onto the space of maximal ideals of \( C(X) \) with the hull-kernel topology. Thus the topology of \( X \) is determined by the ring \( C(X) \) if \( X \) is compact; that is, if \( X_1 \) and \( X_2 \) are compact spaces and the rings \( C(X_1) \) and \( C(X_2) \) are isomorphic, then \( X_1 \) and \( X_2 \) are homeomorphic. In his pioneering paper [Hew], E. Hewitt extended this to a wider class of spaces as follows.

If \( X \) is any (Tychonoff) space and \( M \) is a maximal ideal of \( C(X) \), then \( C/M \) is a totally ordered field containing \( \mathbb{R} \) which is called \textit{real} or \textit{hyper-real} according as \( C/M = \mathbb{R} \) or \( C/M \) contains \( \mathbb{R} \) properly. In the latter case, \( C/M \) is called an H-field in [ACCH], [Dow], and [Ro], and \( M \) is called a \textit{hyper-real ideal}. If every free ideal of \( C(X) \) is hyper-real, then \( X \) is said to be \textit{realcompact}. By now, this terminology is standard, but such spaces are called \( Q \)-spaces in [Hew], \textit{realcomplete} in [AS] and Hewitt - Nachbin complete in [We]. The space of maximal ideals of \( C(X) \) is homeomorphic to \( \beta X \) under a map that leaves \( X \) pointwise fixed, and the subspace \( vX \) of real maximal ideals of \( C(X) \) is called the \textit{realcompactification} of \( X \). Since the space of maximal ideals of \( C(X) \) with the hull-kernel topology is homeomorphic with \( \beta X \), the inclusion \( X \subset vX \subset \beta X \) holds, and it is clear that an isomorphism of \( C(X_1) \) onto \( C(X_2) \).
induces a homeomorphism of $vX_2$ onto $vX_1$. See [GJ1, Chap.8].

This lovely result seems at first to put us into a paradise where we can use ring-theoretic techniques to prove topological theorems and vice versa. The first hiss of the serpent comes from the complicated nature of the class of realcompact spaces and realcompactifications.

The class of realcompact spaces includes all Lindelöf spaces (and hence all separable metric spaces), is closed-hereditary, and productive. Thus every closed subspace of an arbitrary product of copies of $\mathbb{R}$ is realcompact, and the converse holds as well; that is $X$ is realcompact if and only if it is homeomorphic to a closed subspace of a product of real lines; see [GJ1, Chap. 12]. While this characterization of realcompactness is pretty, it is often difficult to apply; say, to answer the question of whether spaces that are metrizable must be realcompact. To answer this question, one must enter the world of axiomatic set theory.

An Ulam measure $\mu$ on a set $S$ is a countably additive measure on $S$ such that (i) $T \subset S$ implies $\mu(T) = 0$ or $\mu(T) = 1$, (ii) $x \in S$ implies $\mu(\{x\}) = 0$, and (iii) $\mu(S) = 1$. An infinite cardinal $\alpha$ is said to be (Ulam) measurable if there is an Ulam measure on a set $S$ of cardinality $\alpha$; otherwise $\alpha$ is said to be nonmeasurable. If $X$ is a discrete space, $M$ is a real ideal of $C(X)$, and $T \subset X$, let $\mu(T) = 1$ if $T \in \mathcal{Z}(M)$, and $\mu(T) = 0$ otherwise. Then $\mu$ is a countably additive measure on $X$ which will be an Ulam measure if and only if $M$ is free. It follows that a discrete space is realcompact if and only if the cardinality of $X$ is nonmeasurable. S. Ulam showed in 1929 that any measurable cardinal has to be tremendously large in the following sense: (i) The smallest infinite cardinal $\aleph_0$ is nonmeasurable, (ii) any cardinal less than a nonmeasurable cardinal is nonmeasurable, (iii) the sum of nonmeasurably
many nonmeasurable cardinals is nonmeasurable, and (iv) if \( \alpha \) is nonmeasurable, so is \( 2^{\alpha} \).

If Zermelo-Fraenkel set theory with the axiom of choice (ZFC) is consistent, so is ZFC with the hypothesis that no cardinal is measurable; see [GJ1, Chap. 12]. Despite this, the possible existence of a measurable cardinal makes the statement of many theorems about realcompactness awkward.

One of the deeper topological characterizations of realcompactness is due to T. Shirotta who showed that \( X \) is realcompact if and only if: (i) \( X \) admits a complete uniform structure compatible with its topology and (ii) every closed discrete subspace of \( X \) has nonmeasurable cardinality; see [GJ1, Chap. 15].

The necessary but ugly addendum (ii) can be replaced with a euphemism in two ways. A discrete space \( X \) is realcompact if and only if it has nonmeasurable cardinality, so (ii) is equivalent to (ii)': every closed discrete subspace of \( X \) is realcompact. Alternatively, in [Ho, p. 12], R. Hodel defines the extent \( e(X) \) of \( X \) to be the maximum of \( \omega \) and \( \sup \{ \text{card} D : D \subseteq X \text{ closed and discrete} \} \). So (ii) is equivalent to (ii)'\(\text{''} e(X) \) is nonmeasurable.

A useful corollary is that every metrizable (indeed, every paracompact) space of nonmeasurable extent is realcompact. This follows also from a more general result given in [PW, 5.11 (1)].

A space \( X \) is said to be \( z \)-embedded in a space \( Y \) if each \( Z \in Z(X) \) is the intersection with \( X \) of some \( Z' \in Z(Y) \). Every \( C^* \)-embedded and every cozero set of a space \( Y \) is \( z \)-embedded in \( Y \) as is every Lindelöf space contained in it. The notion of \( z \)-embedding was introduced by R. L. Blair and many of his results are given in [We, Section 10]. Blair showed also that a countable union of \( z \)-embedded realcompact
spaces is realcompact. Weir attributes the result cited above about Lindelöf spaces being z-embedded in any space containing it to M. Henriksen and D. G. Johnson who took care to attribute it to M. Jerison in [HeJo].

Such nice topological properties as normality and local compactness do not travel well between $X$ and $vX$. Recall that a space $X$ is called pseudocompact if $C(X) = C^*(X)$, or, equivalently, if $vX = \beta X$. The classical Tychonoff plank is pseudocompact and not normal, so the normality of $X$, and the existence of countably compact spaces that fail to be locally compact shows that local compactness need not descend from $vX$ to $X$. Also A. Stone showed that $X$ may be normal without $vX$ being normal, and W. Comfort obtained a similar result for local compactness. See [We, Sections 8 and 10].

If this is not enough to shake your faith in paradise, read on.

2. The big problems: translation, representation, and characterization.

How does one take advantage of the fact that $C(X)$ determines $X$ if $X$ is realcompact? The first step is to be able to translate algebraic statements about $C(X)$ into topological statements about $X$ and vice versa. Sometimes this is easy; e.g., it is an exercise to verify that $f$ is invertible in $C(X)$ if and only if $Z(f)$ is empty, that $f$ fails to be a proper divisor of zero if and only if $\text{Int} \ Z(f)$ is empty, and $X$ is connected if and only if $C(X)$ has only the trivial idempotents $0$ and $1$. (Thus $X$ is connected if and only if $vX$ is connected).

Before giving some other translations that are less obvious, the order properties of $C(X)$ must be discussed. If we let

$$(f \land g)(x) = \min(f(x), g(x)),$$

and
\[(f \lor g)(x) = \max(f(x), g(x))\]

for each \(f, g \in C(X)\) and \(x \in X\), then \(C(X)\) is a lattice whose induced partial order is given by \(f \geq g\) if \(f(x) \geq g(x)\) for all \(x \in X\). Clearly \(f\) and \(g \geq 0\) imply \((f+g) \geq 0\) and \((fg) \geq 0\), so \(C(X)\) is a lattice - ordered ring which is a subdirect product of copies of \(\mathbb{R}\). So \(C(X)\) is an f-ring; see [BK, Chap. 9]. Since \(f \geq 0\) in \(C(X)\) if and only if \(f = g^2\), the order on \(C(X)\) is an algebraic invariant, the lattice structure of \(C(X)\) is determined by its structure as a ring. (A limited converse also holds; see below). A sample list of topological translations of algebraic properties follows.

<table>
<thead>
<tr>
<th>(C(X))</th>
<th>(X)</th>
</tr>
</thead>
<tbody>
<tr>
<td>Every prime ideal of (C(X)) is maximal</td>
<td>(X) is a (P)-space (i.e., every (G_\delta) in (X) is open); see [GJ1, Chap 4].</td>
</tr>
<tr>
<td>Every finitely generated ideal of (C(X)) is principal</td>
<td>(X) is a (F)-space (i.e., every zero set of (X) is (C^*)-embedded); see [GJ1, Chap 14].</td>
</tr>
<tr>
<td>Every finitely generated ideal containing a nondivisor of 0 is principal</td>
<td>(X) is a quasi-(F)-space (i.e., every dense cozero set of (X) is (C^*)-embedded); see [DHH, Sec. 5].</td>
</tr>
<tr>
<td>(C(X)) is a conditionally complete lattice</td>
<td>(X) is extremally disconnected [ED]; (i.e., the closure of every open set is open); see [GJ1, 3N]</td>
</tr>
<tr>
<td>(C(X)) is conditionally (\sigma)-complete as a lattice</td>
<td>(X) is basically disconnected [BD]; (i.e., the closure of every cozero set is open); see [GJ1,3N].</td>
</tr>
</tbody>
</table>

The following diagram of implications holds

\[
P \downarrow
\]

\[
ED \rightarrow BD \rightarrow F \rightarrow quasi-F
\]

and none of these implications hold in reverse; see [GJ1] and [DHH].
RINGS OF CONTINUOUS FUNCTIONS FROM AN ALGEBRAIC POINT OF VIEW

Notice that each of these “nice” algebraic properties translates into a topological property that most topologists consider pathological. This is typical, and the hiss of the serpent gets louder. As noted above, connectedness is equivalent to there being only trivial idempotents, and, within the class of realcompact spaces, compactness of $X$ is equivalent to the existence for each $f$ in $C(X)$ a positive integer $n$ such that $|f| \leq n-1$. As will be noted below, some other “nice” topological properties have useful algebraic translations, but many others, like metrizability and local connectedness seem to lack nonponderous topological translations. It has been shown many times (for the latest such proof see [B]) that $\beta X$ is locally connected if and only if $X$ is locally connected and pseudocompact, (i.e., every continuous function is bounded). So any topological translation of local connectedness in the class of realcompact spaces must involve properties of $C(X)$ that do hold in $C^*(X)$; see [He 1].

Perhaps one of the more valuable uses of rings of continuous functions is as a way of representing archimedean lattice-ordered rings that are algebras over $\mathbb{R}$ with an identity element $e$ that is a weak order unit (i.e., $e \wedge x = 0$ implies $x = 0$). They are called $\Phi-$ algebras and can be represented as algebras of extended real-valued continuous functions as will be seen below.

If $X$ is a compact space, let $\mathcal{D}(X)$ denote the family of continuous functions $f$ on $X$ into the two-point compactification $\gamma = \mathbb{R} \cup \{\pm \infty\}$ of $\mathbb{R}$ such that $\{x \in X : f(x) \in \mathbb{R}\}$ is a dense (open) subset of $X$. If $f \in \mathcal{D}(X)$, let $\mathcal{R}(f) = \{x \in X : f(x) \in \mathbb{R}\}$. If $f, g \in \mathcal{D}(X)$ and the sum of $f$ and $g$ on $\mathcal{R}(f) \cap \mathcal{R}(g)$ has a (necessarily unique) extension over $X$, then we say that $(f + g) \in \mathcal{D}(X)$. Similarly $fg, f \vee g$, and $f \wedge g$ are defined when they are in $\mathcal{D}(X)$. While $\mathcal{D}(X)$ is always a lattice closed under scalar multiplication, it is usually not an algebra. Indeed $\mathcal{D}(X)$ is an algebra if and only if
$X$ is a quasi-F-space; see [HeJo, 2.2]. The elements of $\mathcal{D}(X)$ are called (continuous) extended real-valued functions and it is shown that:

**2.1 Theorem** Every $\Phi$ algebra $A$ is isomorphic to an algebra $\bar{A}$ of extended real-valued functions on a compact space $X$. Moreover if $K_1, K_2$ are disjoint closed subsets of $X$, there is an $\bar{a} \in \bar{A}$ such that $\bar{a}[K_1] = \{0\}$ and $\bar{a}[K_2] = \{1\}$.

Before outlining the proof of this representation theorem, we recall that an l-ideal of a lattice-ordered ring is the kernel of a homomorphism preserving both lattice and the ring operations. Every l-ideal $I$ is a ring ideal such that if $b \in I$ and $|a| \leq |b|$, then $a \in I$. (In [GJ1], an l-ideal is called absolutely convex). Every l-ideal of a $\Phi$-algebra is contained in a maximal l-ideal. If $\mathcal{M}(A)$ is the set of all maximal l-ideals endowed with the hull-kernel topology, then $\mathcal{M}(A)$ is compact. Moreover, for each $M \in \mathcal{M}(A)$, $A/M = \mathbb{R}$ or $A/M$ contains $\mathbb{R}$ properly, and we call $M$ real or hyper-real accordingly. In either case, if $M \in \mathcal{M}(A)$ and $a \geq 0$, then $\bar{a}(M) = \inf \{r \in \mathbb{R} : (a + M) \leq r\}$ is a real number or $+\infty$ (in case $(a + M) > r$ for each $r \in \mathbb{R}$).

Since $A$ is archimedean, $\cap\{M \in \mathcal{M}(A) : \bar{a}(M) \in \mathbb{R}\} = \{0\}$, so $\bar{a} : \mathcal{M} \to \mathbb{R} \cup \{+\infty\}$ is real-valued on a dense subset of $\mathcal{M}(A)$, and turns out to be continuous as well. If $a \in A$ is arbitrary and $M \in \mathcal{M}(A)$, let $\bar{a}(M) = \bar{a}^+(M) - \bar{a}^-(M)$, where $a^+ = a \vee 0$ and $a^- = (-a) \vee 0$. Since $a^+ \wedge a^- = 0$, $\bar{a}$ is a continuous extended real-valued function on $\mathcal{M}(A)$ and the map $a \to \bar{a}$ is the desired isomorphism.

This result was improved upon by D. Johnson and J. Kist, and both results may be derived from the Yosida representation Theorem; see [LZ; Chapter 4].

This representation theorem has been used to characterize $C(X)$ algebraically within the class of $\Phi$-algebras in an internal way. A $\Phi$-algebra $A$ is said to be uniformly closed if its image in $\mathcal{D}(\mathcal{M}(A))$ is closed under uniform convergence. (This is an
algebraic concept since closure under uniform convergence is equivalent to saying that every Cauchy sequence of elements of $A$ converges). It is noted in [HeJo] that $A$ is uniformly closed if and only if $\tilde{A}$ is an order-convex subalgebra of $D(M(A))$ containing $C(M(A))$. The $\Phi$-algebra $A$ is called a $\Phi$-algebra of real-valued functions if $\cap R(A) = \{0\}$, where $R(A)$ denotes the set of real maximal $l$-ideals of $A$. If $a \in A$, let $Z(a) = \{ M \in M(A) : |a| + M \leq \frac{1}{n}e + M \text{ for } n = 1, 2, \ldots \}$. $A$ is said to be closed under inversion if for each $a \in A$ the smallest $l$-ideal containing $a$ is all of $A$ whenever $Z(a) \cap R(A) = \emptyset$. The following lemma is usually at the crux of most characterization theorems; see [HeJo, 5.2].

2.2 Lemma A $\Phi$-algebra $A$ is isomorphic to $C(Y)$ for some Tychonoff space $Y$ if and only if

(i) $A$ is an algebra of real-valued functions,

(ii) $A$ is uniformly closed,

(iii) $A$ is closed under inversion, and

(iv) $R(A)$ is $z$-embedded in $M(A)$.

While conditions (i), (ii), and (iii) are algebraic, and insure that $A$ is isomorphic to an order-convex subalgebra of $C(R(A))$, condition (iv) is doubly unsatisfactory from our point of view. It not only fails to be algebraic in character, but it refers to properties of the external object $C(R(A))$.

This lemma or variations on it may be used easily to characterize $C(X)$ in case $X$ is compact (replace (iv) by requiring that if $a \in A$, then $|a| \leq n \cdot e$ for some positive integer $n$), and in case $X$ is locally compact and $\sigma$-compact (replace (iv) by requiring that $R(A) = R(h)$ for some $h \in A$). Characterizations for other classes of (realcompact) Tychonoff spaces are mentioned in [HeJo]. There is a characterization
of $C(X)$ for $X$ for any realcompact $X$ due to F. Anderson [Ande] and corrected by G. Jensen in [Jen], but no algebraist I know can imagine verifying its complicated hypothesis in any concrete instance. Many other characterizations have been given, too numerous to mention in toto. Having many more beauties to choose from than did Paris, I single out a few as particularly worthy.

The first is due to D. Plank. He calls a $\Phi$-algebra normal if each of its $l$-ideals that is closed under uniform convergence is contained in a real maximal $l$-ideal. In [Pl], he establishes the following lovely result.

2.3 Theorem (Plank) A $\Phi$-algebra $A$ is isomorphic to $C(Y)$ for some Lindelöf space $Y$ if and only if $A$ is a uniformly closed $\Phi$-algebra of real-valued functions that is closed under inversion and normal.

Part of the reason that Plank's Theorem holds is that fact that a Lindelöf space is $z$-embedded in any Tychonoff space containing it (See Section 1).

To see how for condition (i), (ii), and (iii) of Lemma 2.2 fall short of characterizing a $C(X)$, in the general case, observed as in [HeJo] that the $\Phi$-algebra $B$ of Baire functions on $\mathbb{R}$ into $\mathbb{R}$ satisfies (i), (ii), and (iii) of Lemma 2.2, while $\mathcal{R}(B)$ is the set $\mathbb{R}$ with the discrete topology. Thus, while $B$ has power $c$, the algebra $C(\mathcal{R}(B))$ has power $2^c$.

In 1975, A. Hager attempted to characterize $C(Y)$ for an arbitrary realcompact space $Y$ in a succinct way in which will be described next. It remains doubtful that he succeeded, but his attempt has great value.

An ideal $D$ of a $\Phi$-algebra $A$ is called dense if its only annihilator is 0. For any subset $S$ of $A$, let $\text{coz } S = \bigcup \{\text{coz } \tilde{a} : \tilde{a} \in S\}$, where $a \rightarrow \tilde{a}$ is the map of $A$ into $\mathcal{D}(\mathcal{M}(A))$ defined above. Note the ideal $D$ of $A$ is dense if and only if $\mathcal{R}(A) \subset \text{coz } D$. 
A dense ideal is said to be \( R \)-dense if it is contained in no real maximal ideal. If, for each \( R \)-dense ideal \( D \) of \( A \), every module homomorphism \( \varphi \in \text{Hom}_A(D, A) \) is given by multiplication by some fixed element of \( D \), the \( \Phi \)-algebra \( A \) of real-valued functions is said to be closed under strong inversion. \( A \) may be regarded as a subalgebra of \( C(\mathcal{R}(A)) \), and \( A \) is closed under strong inversion if and only if whenever for \( a, b \in A \), the quotient of \( a/b \) is defined on a neighborhood of each \( x \in \mathcal{R}(A) \), we have \( a/b \in A \).

Modifying Hager's notation slightly, we call a \( \Phi \)-algebra that satisfies (i) and (ii) of Lemma 2.2 and is closed under strong inversion an \( SI \)-algebra. A. Hager made the following conjecture as long ago as 1967. In [Ha 2] he calls it a "working conjecture" and expresses doubt about its validity.

2.4 Conjecture (Hager) A \( \Phi \)-algebra \( A \) is isomorphic to \( C(X) \) for some realcompact space \( X \) if and only if \( A \) is an \( SI \)-algebra.

By way of evidence in its favor, Hager showed that every \( C(X) \) is an \( SI \)-algebra. Moreover if \( A = C(X) \) for some realcompact \( X \), then we may assume \( X = \mathcal{R}(A) \). A space \( X \) is said to be \( G_\delta \)-closed in a space \( Y \) if for each \( y \in Y \), there is a \( G_\delta \) -set (or, equivalently a zeroclass) of \( Y \) containing \( y \) and missing \( X \). Note that \( X \) is real compact if and only if it is \( G_\delta \)-closed in \( \beta X \). To facilitate the rest of the discussion about Hager's conjecture, I introduce the following definition.

2.5 Definition A compactification \( K \) of a space \( X \) is called large if \( K = \beta V \) whenever \( V \) is an open neighborhood of \( X \) in \( K \).

Clearly \( \beta X \) is a large compactification of \( X \) and is clearly the only large compactification of \( X \) if \( X \) is locally compact or Lindelöf. Consider the following statements

(*) If \( K \) is a large compactification of \( X \), then \( K = \beta X \), and

(**) If \( K \) is a large compactification of \( X \) and if \( X \) is \( G_\delta \)-closed in \( K \), then \( K = \beta X \).
According to M. A. Sola, Hager conjectured that (*) holds as early as 1970, and in [Hal, 7,2], Hager shows that 2.4 holds if and only if (***) holds. It is doubtful that Hager conjectured (*), but it is still an interesting question.

In his University of South Carolina doctoral dissertation [S, Chap. 5] Sola shows that (*) need not hold with an example that does not disprove (**), so Hager's conjecture in 2.4 remains open.

What Sola does is to show that Prabir Roy's example $\Delta$ of a zero-dimensional metrizable space of power $c$ for which $\beta \Delta$ fails to be zero dimensional has as a large compactification the maximal zero dimensional compactification $\zeta \Delta \neq \beta \Delta$. (The space $\zeta X$ may be regarded as a space of ultrafilters on the collection of clopen subsets of $\Delta$; see, for example [PW, Section 4.7], where $\zeta \Delta$ is denoted by $\beta \Delta$). Roy's example was announced first in [Roy] in 1962. It is complicated in nature; for a relatively easy development of its properties, see [Pe, Sect 7.4] (where $\Delta$ is denoted by $P$).

As Sola also observes, $\Delta$ is not $G_\delta$-closed in the large compactification $\zeta \Delta$. For, as was shown by P. Nykos in 1971, (see [Ny]), $\Delta$ fails to be $N$-compact (i.e., $\Delta$ cannot be embedded as a closed subspace of a product of copies of the countable discrete space $N$) and it follows that there is a space $N\Delta$ properly in between $\Delta$ and $\zeta \Delta$ such that every $G_\delta$ of $\zeta \Delta$ containing a point of $N\Delta \setminus \Delta$ meets $\Delta$. ($N\Delta$ is the $N$-compactification of $\Delta$; see [Ny] or [S, Chapter 4]).

Thus Hager's conjecture remains unresolved. Most people who have worked on the problem (including Hager) guess that it is false. If so, a counterexample will be hard to find. The space $\Delta$, being metrizable of nonmeasurable cardinal is realcompact – as will be any counterexample to (**).

Is the class of spaces for which (***) holds productive and/or closed hereditary- as is
the class of realcompact spaces? Can \( (**) \) be used as a unifying principle from which characterizations of \( C(X) \) for several classes of realcompact spaces can be derived fairly easily?

Surely, Hager’s work in [Ha 1] should be exploited further.

In the same spirit, I mention G. DeMarco’s characterization of \( C(X) \) for \( X \) paracompact or strongly paracompact [DeM]. Recall that \( X \) is paracompact (resp. strongly paracompact) if every open cover of \( X \) has an open locally finite (resp. star finite) open refinement. A locally Lindelöf space is strongly paracompact if and only if it is a topological sum of Lindelöf spaces, but there are strongly paracompact spaces that fail to be locally Lindelöf. De Marco shows that a \( \Phi \)-algebra \( A \) is isomorphic to a \( C(X) \) for some strongly paracompact realcompact space if and only if \( A \) is an \( SI \)-algebra and each \( R \)-dense ideal of \( A \) contains an \( R \)-dense ideal that is a projective \( A \)-module. The reader is referred to [DeM] for the technically more complicated characterization in the paracompact case.

In a study related to the problem of characterizing \( C(X) \), A. Hager and D. Johnson call a uniformly closed subalgebra of \( C(X) \) that separates points from disjoint closed sets, contains all constant functions, and is closed under inversion an ALGEBRA ON \( X \). (The capitalization is used here to avoid confusion with the terminology used above; Hager and Johnson stick to lower case). They show in [Ha Jo] that if \( vX \) is Lindelöf, then any ALGEBRA on \( X \) is a \( C(Y) \) for some \( Y \) and they asked if the converse holds. In [Bl 1] and [Bl 2], J. Blasco shows that it does if either \( X \) is a topological group or \( X \) is paracompact. In the general case, their question remains open for a more detailed study of the structure of ALGEBRAS, see [Ha 2].

I close this section with a mention of another use of algebraic properties of \( C(X) \)
to obtain information about \( X \). A space \( X \) is called \textit{rimcompact} if each point of \( X \) has a base of neighborhoods with compact boundaries. Each rimcompact space has a compactification \( \Phi X \) such that

(a) \( \Phi X \setminus X \) is zero-dimensional, and

(b) if \( \Psi X \) is any compactification of \( X \) such that \( \Psi X \setminus X \) is zero-dimensional, then there is a continuous map of \( \Phi X \) onto \( \Psi X \) that extends the identity map of \( X \) onto \( X \).

The space \( \Phi X \) is called the \textit{Freudenthal compactification} of \( X \) and is essentially unique.

Let \( C^\#(X) = \{ f \in C(X) : \text{for each } M \in \mathcal{M}(C(X)), \text{there is an } r \in \mathbb{R} \text{ with } (f - r) \in M \} \). The subalgebra \( C^\#(X) \) is a subalgebra of \( C^*(X) \) and consists only of closed mappings of \( X \) into \( \mathbb{R} \) if \( X \) is rimcompact and realcompact. In different notation it was introduced by N. Shilkret in [Sh] who called it the \textit{Gelfand subalgebra}, and independently L. Nel and Riordan in [NR]. In [He 2], I showed that if \( C^\#(X) \) is rich enough to separate points from disjoint closed sets, i.e., to determine a compactification of \( X \), and \( X \) is realcompact, then \( X \) is rimcompact and the compactification is \( \Phi X \), and I include an example due to S. Willard to show that \( C^\#(X) \) need not determine a compactification of \( X \) even if \( X \) is both rimcompact and realcompact. The question of exactly when \( C^\#(X) \) determines \( \Phi X \) remains open. One may also ask if some related subalgebra of \( C(X) \) may be found which determines \( \Phi X \) whenever \( X \) is both rimcompact and realcompact.

There is a surprisngly large literature on this seemingly arcane area. It is summarized and improved upon by R. André in his University of Manitoba master’s Thesis [André]. See also the Dickman-McCoy monograph on the Freudenthal compactification.
and the references therein [DM]. Also, in [Dom], J. Domínguez studies the Gelfand subalgebra for continuous functions with values in a nonarchimedean field.

3. H-fields and residue class fields of prime ideals: the really tough problems

The structure of H-fields has been studied a lot in the last 40 years. Much has been learned, but there are still many open problems. The latter seem to be quite difficult and seem to require heavy use of set theory, model theory and study of the structure of ultrafilters for their solution.

In [GJ1, Chap 13], it is shown that for every maximal ideal \( M \) of \( C(X) \), the residue class field \( C(X)/M \) is real-closed. Since much of what we know about an H-field depends heavily on its structure as an ordered set, I pause to review some definitions.

A cardinal number \( \aleph_\alpha \) is regular if \( \omega_\alpha \) contains no cofinal subset of smaller cardinality; otherwise it is singular. If \( \aleph_\alpha \) is regular, then \( \alpha = \beta + 1 \) for some cardinal \( \beta \). More generally, the least cardinal number of a cofinal subset of a cardinal \( \beta \) is denoted by \( cf(\beta) \). Thus, \( \beta \) is regular if and only if \( cf(\beta) = \beta \). If \( A \) and \( B \) are subsets of a (totally) ordered set \( L \), and \( a \in A \) and \( b \in B \) imply \( a < b \), we write \( A < B \).

3.1 Definition Suppose \( L \) is an ordered set and \( \alpha \) is an ordinal number. If, whenever \( A \) and \( B \) are subsets of \( L \) of cardinality less than \( \aleph_\alpha \) such that \( A < B \), there is an \( x \) in \( L \) such that \( A < x < B \), the set \( L \) is called an \( \eta_\alpha \)-set.

Since \( A \) or \( B \) may be empty, an \( \eta_\alpha \)-set has no cofinal or coinitial subset of power smaller than \( \aleph_\alpha \). The ordered set \( Q \) of rational numbers is the only \( \eta_0 \)-set up to an order isomorphism; and it is known that any ordered set of power \( \aleph_\alpha \) may be embedded in an \( \eta_\alpha \)-set; see [GJ1 130].
This section depends heavily on the theory of ultrafilters and [CN] is a principal reference, so it must be pointed out that my terminology conflicts with theirs; what I call an \( \eta_\alpha \)-set is called an \( \eta_\omega \)-set in [CN].

In [EGH] and [GJ1, Chap. 13], it is shown that every H-field is an \( \eta_1 \)-set. An ordered field that is an \( \eta_\alpha \)-set is called an \( \eta_\alpha \)-field. In this language, every H-field is a real-closed \( \eta_1 \)-field. It was shown also in [EGH] that any two real-closed \( \eta_\alpha \)-fields of cardinality \( \aleph_\alpha \) are isomorphic if \( \alpha > 0 \). It is noted that there are no \( \eta_\alpha \)-sets of power \( \aleph_\alpha \) if \( \aleph_\alpha \) is singular, and if there is an \( \eta_{\alpha+1} \)-set of cardinality \( \aleph_{\alpha+1} \), then \( 2^{\aleph_\alpha} = \aleph_{\alpha+1} \). This implies that if CH holds, then two real-closed fields of power \( c \) are isomorphic. Jumping a bit ahead in our story, A. Dow in [Dow] (improving on consistency results in [ACCH] and [Ro]) showed that the converse holds as well; if CH fails, then there is a pair of nonisomorphic H-fields of power \( c \).

An H-field that is a homomorphic image of \( C(D) \) where \( D \) is a discrete space is called an ultrapower of \( \mathbb{R} \). In [EGH] it is shown that there are ultrapowers of arbitrarily large cardinality. Indeed, all of the H-fields exhibited in [EGH] are ultrapowers, as are the H-fields exhibited by A. Dow in [Dow, 2.3] in the absence of \( CH \). Dow's H-fields are homomorphic images of \( C(N) \).

The results in [EGH] led to a variety of questions only some of which were posed explicitly in that paper.

If it were true that every H-field is an \( \eta_\alpha \)-field of power \( 2^{\aleph_\alpha} \) for some \( \alpha > 0 \), then each H-field would be determined by its cardinal number at least if GCH (the generalized continuum hypothesis) holds. It is asked in [EGH, 5.2] if this latter is true. Note that its truth would imply that there is no H-field of singular cardinality.

It is not obvious that there are any real-closed \( \eta_\alpha \)-fields for any \( \alpha > 1 \), and this is
listed as an open problem in [GJ1, 130]. In a paper that has largely been forgotten
by workers on H-fields, N. Alling constructed in his 1962 paper [Al 1] for each regular
cardinal \( \aleph_{\alpha+1} \) a real-closed \( \eta_{\alpha+1} \)-field of power \( \aleph_{\alpha+1} \) under the necessary assumption
that \( 2^{\aleph_\alpha} = \aleph_{\alpha+1} \). In another not well-known expository paper present in Rome, Italy
in 1973, [Al 2] Alling summarized the state of knowledge about H-fields at that time.
He reported that Keisler had shown if GCH hold, and \( \alpha > 0 \), then every real-closed
\( \eta_\alpha \)-field of power \( \aleph_{\alpha+1} \) is an ultrapower, but not every ultrapower takes this form.
Keisler’s results in [Ke] were phrased in the language of model theory and hence
had not been read by many workers in \( C(X) \). In particular, \( \eta_\alpha \)-fields are called
\( \aleph_\alpha \)-\textit{saturated fields}, so the applicable results even in [CN, pp. 320-325] are hard to
recognize.

A massive attack by four very capable mathematicians on the problems of deter-
mining the structure of H-fields was made in the mid-1970’s. While some of their
results were presented at the Prague topology conference in 1976, [ACCH] did not
appear until 1981. Below, I list some of its major results.

3.2 \( m \) is the cardinal number of an H-field if and only if \( m = m^{\aleph_\alpha} \).

If a fragment of GCH holds, then it follows from [Jec, Corollary 2, p. 49] that \( \aleph_\omega \)
is a singular cardinal satisfying 3.2. As pointed out to me by W. Comfort, to get an
example in ZFC, it is enough to take a beth cardinal that is a strong limit cardinal
of uncountable cofinality; see [ACCH, Sect 1] or [CN, Chap. 1] for the appropriate
definitions.

In [ACCH, 7.7], it is shown that every H-field is a homomorphic image of \( C(\mathbb{R}^\Delta) \)
for some cardinal \( \Delta \), and, assuming GCH, a cardinal number is exhibited that satisfies
the conditions of 3.2 above that fails to be the cardinal number of any ultrapower;
loc. cit., 5.11.

If $M$ is a hyper real ideal of $C(X)$, let $m$ denote the least cardinal number of a zero set in $\mathcal{Z}(M)$. Must the cardinality of $C/M$ exceed $m$? Having answered this in the affirmative if $m = \aleph_0$ or $c$, or if $X$ is discrete, the problem is posed in 5.3 of [EGH], and is solved in the negative in ZFC in [ACCH, Section 6], where it is pointed out that 5.3 had been solved earlier by a variety of authors under set theoretic assumptions.

In 5.4 of [EGH] the reader is asked to investigate to what extent results on H-fields depend on CH. Dow answers one such question definitively in [Dow] as noted above, and in [ACCH] many more problems of this sort are considered.

[ACCH] is full of other interesting results which the reader is urged to examine in the original. These authors also pose many intriguing questions. I repeat only three of them.

3.3 Is every real-closed $\eta_1$-field an H-field?

3.4 Are order isomorphic H-fields algebraically isomorphic?

3.5 Is there any way to connect the cardinal number of the H-field $C(X)/M$ with topological or combinatorial properties of $X$ and $\mathcal{Z}(M)$ respectively?

I hope that this summary serves to generate more interest in the study of H-fields. Success in this area will require expertise in set theory, model theory, topology, and algebra. Perhaps a lot of team efforts will be needed.

To facilitate the ideas presented in the rest of this section, I introduce the following definition

3.6 Definition If $X$ is a topological space and $P$ is a prime ideal of $C(X)$ such that $C(X)/P$ contains $\mathbb{R}$ properly, then $C(X)/P$ is called an H-domain.

While this terminology appears here for the first time, the study of H-domains
is far from new. Every $H$-domain $D$ is a totally ordered integral domain whose field of quotients is real-closed since every positive element has a square root and every monic polynomial of odd degree has a zero in $D$. (This was observed by N. Alling in 1963 in [Al3]). $C(X)/P$ is a $H$-domain as long as $P$ is a nonmaximal prime ideal of $C(X)$, and the set of prime ideals of $C(X)/P$ is totally ordered; it is $\{Q/P : Q$ prime and $Q \supset P\}$. C. Kohls studied prime ideals of $C(X)$ and the order structure of $H$-domains in his Purdue doctoral dissertation which is most of the content of [GJ1, Chap 14]; see also [Ko 1, 2, 3, 4]. In these studies, as in [M 1,2], the emphasis is more on prime ideals than on the algebraic structure of $H$-domains. Until recently, I have seen very few studies of the structure of $H$-domains that are not fields. One worthy of mention is [GJ 2] where some necessary and some sufficient conditions are given in order that the quotient field of an $H$-domain be a $H$-field. It rests heavily on the contents of [GJ1, Chap. 14].

In the last few years two substantial studies of $H$-domains have been made. In [CD 1], a (commutative) integral domain $A$ is called a real-closed ring if it is an ordered ring satisfying the intermediate value property for polynomials. That is, if $a, b \in A, a < b$, and $f(X) \in A[x]$ satisfies $f(a)f(b) < 0$, then there is a $c \in A$ in the open interval $(a, b)$ such that $f(c) = 0$. A prime ideal $P$ of $C(X)$ is called a real-closed ideal if $C(X)/P$ is a real-closed ring. A commutative totally ordered integral domain $A$ is a real-closed ring if and only if $0 < a < b$ implies $a$ divides $b$; that is if and only if $A$ is a valuation ring (i.e., of any two elements, one must be divisible by the other). It turns out that if $P$ is a real-closed ideal, and $Q$ is a prime ideal containing $P$, then $Q$ is a real-closed [CD 1, Prop. 7]. Since every minimal prime ideal of $C(X)$ is a z-ideal, it suffices to determine when z-ideals are real-closed. A technical topological
characterization of real-closed z-ideals which proves to be a useful tool is given in [CD 1, Theorem 1 and Prop. 5]. A few of their results follow:

If $X$ is an $F$-space, then every prime ideal is real-closed, but the converse fails. If $X$ is metrizable, $p \in X$, and $M_p = \{f \in C(X) : f(p) = 0\}$ contains a nonmaximal real-closed ideal, then there is a P-point of $\beta N - N$, i.e., a point in the interior of any $G_\delta$ containing it. It is known that there are models of set theory in which $\beta N - N$ has no P-point and Martin’s Axiom implies their existence; see [vM]. A detailed study of the relationship between the real-closed ideals of the ring $C(N^*)$ of convergent sequences of real numbers and the P-points of $\beta N - N$ is made, and the set of real-closed ideals of $C(D^*)$ (where $D^*$ is the one-point compactification of an uncountable discrete space) and of $C[0,1]$ is studied carefully. Many unsolved problems remain. Many of the questions asked about H-fields may also be asked about H-domains that are real-closed rings. For example, does the order type of such a ring determine it to within an algebraic isomorphism?

[CD 2] is concerned mainly with model-theoretic questions about real-closed rings. In his Rutgers doctoral dissertation, J. Moloney showed that up to isomorphism, there are exactly 10 H-domains of $C(N^*)$. CH is assumed freely in his work, and to the best of my knowledge, nobody has examined whether these results hold with less restrictive set-theoretic assumptions. He also classifies some of the domains that are homomorphic images of $C^\infty([0,1])$. These results are available in preprint form [Mo].

4. Miscellania

This section is about a few topics near the boundary of the scope of this paper which merit at least brief mention.

1. Discontinuous homomorphisms of commutative Banach algebras
If $X$ is a compact space, and we let $\| f \| = \sup \{ | f(x) | : x \in X \}$ for each $f \in C(X)$, then $(C(X), \| \cdot \|)$ is a commutative semisimple Banach algebra. In 1949, I. Kaplansky asked if there could be a norm $\| \cdot \|$ on a $C(X)$ with respect to which it becomes a normed algebra (satisfying $\| fg \|' \leq \| f \|' \| g \|'$), with respect to which $C(X)$ fails to be complete. This problem reduces to the questions of whether there is a discontinuous homomorphism of $(C(X), \| \cdot \|)$ into a Banach algebra. It was 25 years before this question could be answered for any infinite compact $X$, and then, independently, G. Dales and J. Esterle showed that if CH holds and $X$ is a compact space then $C(X)$ has a discontinuous homomorphism. Shortly thereafter, R. Solovay and his student, H. Woodin, announced that there are models for set theory in which whenever $X$ is compact, every homomorphism of $C(X)$ into a Banach algebra is continuous. In a recent book [DW], Dales and Woodin provide an excellent exposition of the Solovay-Woodin independence proofs making these results accessible for the first time to mathematicians that are not experts in model theory. If you want to learn how to use Martin’s Axiom and forcing, this is a great place to start. It also contains a complete set of references and a history of the problem.

II. Other algebraic structures on $C(X)$

Probably many of you are more familiar with the structure of $C(X)$ as a Riesz space (alias vector-lattice) than as an algebra. Indeed, an archimedean Riesz space with a weak order $e$ admits at most one multiplication making it into a $\Phi$-algebra in which $e$ is the identity element. More generally, it always has a unique embedding into a $\Phi$-algebra. The nature of the embedding and the way in which a change of the weak order unit can effect the multiplicative structure is described in [HR 1,2]. This work overlaps with earlier work of P. Conrad on how the multiplication of an f-ring
is determined by its structure as an abelian l-group (not necessarily uniquely in the
nonarchimedean case); see [Co].

In 1956, making use of T. Shirota’s generalization of earlier work of I. Kaplansky
and A. Milgram, I thought I had shown that \( C(X) \) either as a lattice or as a mul-
tiplicative semigroup determined \( vX \); see [He 3]. In [Cs] A. Csazar points out that
there is a gap in Shirota’s reasoning and provides a much simpler way of showing
that \( C(X) \) as a multiplicative semigroup determines \( X \). Indeed, he finds that \( vX \)
is determined by semigroup structures that are much simpler.

Despite this, I find myself more comfortable working with the ring structure in
preference to that of Riesz spaces, lattices, or multiplicative semigroups. That may,
however, just be a personal prejudice.

III. Spaces of Minimal Prime Ideals

Since 1965, there has been a lot written on the space of minimal prime ideals
of a commutative ring with the hull-kernel topology. In case the ring is a \( C(X) \) the
space \( mX \) of minimal prime ideals of \( C(X) \) had been known to be countably compact
since 1965 and was known to be basically disconnected if it is locally compact; see
[He Je] and [Ki]. Recently, in two papers [DHKV1,2], it has been shown that \( mX \)
is always \( \omega \)-bounded, indeed that weakly Lindelöf subspaces of \( mX \) have compact
closures and are \( C^* \)-embedded in \( mX \), but \( mX \) need not be basically disconnected,
or even a quasi-F-space. It is not known exactly when \( mX \) is basically disconnected,
and it would be nice to get a topological characterization on \( mX \).

IV. Covers and completions

In 1958, A. Gleason showed that the projective objects in the category of compact
spaces and continuous maps are extremally disconnected, and there is for each \( X \)
in this category a compact extremally disconnected space minimal with respect to mapping continuously onto $X$. By now most authors call this space the *absolute* of $X$ and denote it by $EX$. The ring $C(EX)$ turns out to be the Dedekind-MacNeille completion of $C(X)$.

Parts of this theory can be extended to the category of Tychonoff spaces and perfect maps (A map is *perfect* if it closed, continuous, and the inverse images of each point is compact), but in general the Dedekind completion of a $C(X)$ need not be a $C(Y)$; see [MJ].

Other kinds of completions of a $C(X)$ may be obtained in similar ways, but each different kind of completion has its own complications. Those who would like to learn more may examine [DHH], [Ha3], [HVW] and [Z]. Excellent expositions of the theory of absolutes are given in [Wa] and [PW].

The Henriksen-Johnson representation Theorem 2.1 above has been used to extend many results about $C(X)$ to archimedean $\Phi$-algebras in appropriately modified form. This could be the subject of another talk at least as long.

I hope I have convinced you that while working in this area will not put you in a mathematical paradise, it can fulfill your needs for working on challenging and interesting problems.
References


[DeM] G. De Marco, *A Characterization of C(X) for X strongly paracompact (or paracompact)*, Symposia Math. 21 (1977), 547-554.


[G] L. Gillman, *Rings of continuous functions are rings*, Rings of Continuous Func-


[HR 1] A. Hager and L. Robertson, Representing and ringifying a Riesz space Sym-
posia math. 21 (1977), 411-430.


